ECON33100 Theory of Income II

Feng Lin*

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*This note is mostly a reorganized version of Professor Mikhail Golosov's lecture note, with additional derivations and reference to some other materials occasionally. It is intended to be shared with students taking the same class, and should be used as if it is distributed under GPL-3.0-or-later; for instance, if this document is modified and shared with future students, the complete source code of the modified version should be made available. There could be errors in this note. Contact: fenglin2@uchicago.edu.

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1 Kaldor Facts and Balanced Growth

1.1 Kaldor Facts

1. Output per capita grows at a constant rate
2. Capital-output ratio is roughly constant
3. Interest rate is roughly constant
4. Distribution of income between capital and labor is roughly constant

1.2 Model Implications

Aggregate production function for the unique final good is

$$
\begin{aligned}
Y(t) &= \tilde{F}(K(t), L(t), \tilde{A}(t)) \\
\dot{K}(t) &= Y(t) - C(t) - \delta K(t)
\end{aligned}
$$

where $\tilde{F}$ is CRS in K, L.

Constant growth of a variable means

$$\frac{\dot{X}}{X} = g_X.$$

We assume that

$$\frac{\dot{Y}}{Y} = g_Y > 0,\ \frac{\dot{K}}{K} = g_K > 0,\ \frac{\dot{C}}{C} = g_C > 0,\ \frac{\dot{L}}{L} = n > 0.$$

Theorem 1.1. *Uzawa*
Constant growth + CRS implies (Uzawa Assumptions)

1. Balanced growth: $g_Y = g_c = g_K \equiv g$
2. Labor augmenting technical change: $\tilde{F}$ can be represented as

   $$\tilde{F} = F(K(t), A(t)L(t))$$

   for some CRS F with $\frac{\dot{A}}{A} = g - n$.
   - Either purely labor augmenting
   - or elasticity of substitution between capital and labor must be 1.

Proof. Note that positive growth rates are part of the constant growth assumption. □

Theorem 1.2. *Perfect Competition*
Suppose, in addition, firms are perfectly competitive

- Monopolistic competition with constant markups would work also.

Uzawa assumptions + constant factor shares implies

1. Interest rate is constant: $R(t) = R^*$.
2. Wages grow at the rate of technology change: $\frac{\dot{w}}{w} = g_A$.

Proof.

$$\alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t))}$$

$$w(t) \equiv \frac{Y(t) - R(t)K(t)}{L(t)}$$

□

Theorem 1.3. *Implication for Preferences*
Suppose utility is $U(C)$ and ρ is a discount factor. The consumer solves the following problem:

$$\max_C \int e^{-\rho t} U(C) \quad \text{s.t. } \dot{K}(t) = R(t)K(t) + w(t)L(t) - C(t) - \delta K(t).$$

The Euler equation can then be written as

$$\frac{\dot{C}}{C} = \frac{1}{\sigma(C)}(R - \delta - \rho),$$

where

$$\sigma(C) \equiv -\frac{U''(C)C}{U'(C)}.$$

Constant interest rates and balanced growth implies that $\sigma(C)$ is constant, i.e.

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma} + const.$$

1.3 Neoclassicial Growth

Infinitely lived representative household with preferences

$$\int e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$$

and inelastic labor supply (for now).

Perfectly competitive firms with CRS technology

$$Y = F(K, AL).$$

Feasibility

$$C + \dot{K} = Y - \delta K,$$
$$L = 1.$$

Renormalize everything by AL, then we can see that this model is isomorphic to neoclassical growth model without growth, and therefore we know that

- Competitive equilibrium is efficient.
- k, c, y converge to the steady state.

Theorem 1.4. *Growth Model and Kaldor Facts*
Steady state of the neoclassical growth model is consistent with Kaldor facts.

1.4 Other Observations

1.4.1 Endogenizing Labor

Balanced growth path preferences with labor supply

$$U(C, L) = \begin{cases} \frac{C^{1-\sigma}}{1-\sigma} v(L) & \sigma \neq 1 \\ \ln(C) + v(L) & \sigma = 1 \end{cases}.$$

1.4.2 Other

Balanced growth requires

- either no technical progress for capital
- or unit elasticity of substitution between capital and labor

Prices of captial goods fell dramatically $\Rightarrow$ suggests some capital-augmenting technical change.

2 Structural Change: Demand Side

2.1 Kuznets Facts

Sector	Employment Share	Consumption Share
Agriculture	declines	declines
Manufacturing	stable	stable
Services	increases	increases

Aim to reconcile broad structural changes emphasized by Kuznets with simultaneous constancy of aggregate variables emphasized by Kaldor.

- Preference-driven
- Technology-driven

2.2 Model Setup

2.2.1 Household

Infinitely living representative household with exogenous labor supply $L = 1$ (i.e. HH does not choose L).

Preferences:

$$U \equiv \int_{[0,\infty)} e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} dt,$$

where

$$c = \left(c^A - \gamma^A\right)^{\eta^A} \left(c^M\right)^{\eta^M} \left(c^S + \gamma^S\right)^{\eta^S}, \quad \eta^A + \eta^M + \eta^S = 1.$$

Budget constraint:

$$\sum p^i c^i + \dot{K} = wL + (r - \delta)K.$$

We normalize $p^M = 1$.

- Stone-Garry preferences
  - Minimum or subsistence level of agricultural (food) consumption γ^A
  - After γ^A has been achieved, household starts to demand other items
- γ^S: household will spend on services only after certain levels of agricultural and manufacturing consumption have been reached.
- Highly tractable: generate linear demand functions
  - Agriculture share decreases in income
  - Services share increases in income

2.2.2 Firm

Production functions for $i \in \{A, M, S\}$:

$$Y^i = B^i F\left(K^i, XL^i\right).$$

Note that the technology progress is the same across sectors.

F satisfies usual neoclassical assumptions, and constant rate of growth for X

$$\frac{\dot{X}}{X} = g.$$

Firm's problem is

$$\max_{Y^i, K^i, L^i} p^i Y^i - wL^i - rK^i$$
$$\text{s.t. } Y^i = B^i F\left(K^i, XL^i\right).$$

2.2.3 Market Clearing

Labor and Capital

$$K^A + K^M + K^S = K, \quad L^A + L^M + L^S = L = 1.$$

Manufacturing good is used in production of investment good

$$I + c^M = Y^M, \quad \dot{K} = I - \delta K.$$

Agriculture and Service Goods

$$c^A = Y^A, \quad c^S = Y^S.$$

2.2.4 Competitive Equilibrium

Given initial K_0, collection of prices and quantities, such that

- consumers choose their quantities optimally given prices
- firms choose their quantities optimally given prices
- all markets clear

2.3 Optimality Conditions

2.3.1 Household

The Hamiltonian is

$$H = \frac{c^{1-\theta} - 1}{1 - \theta} + \lambda \left(w + (r - \delta)K - \sum p^i c^i\right).$$

The FOCs are

$$\begin{aligned}
\frac{\partial H}{\partial c^i} =& c^{-\theta}\frac{c}{c^i-\gamma^i}\eta^i - \lambda p^i = 0, && [c^i]\\
-\frac{\partial H}{\partial K} =& -\lambda(r-\delta) = \dot{\lambda} + \rho\lambda, && [K]\\
\frac{\partial H}{\partial \lambda} =& w + (r-\delta)K - \sum p^i c^i = \dot{K} && [\lambda].
\end{aligned}$$

2.3.2 Firm

Capital

$$p^i B^i F_K(K^i, XL^i) = r.$$

Labor

$$p^i B^i F_L(K^i, XL^i)X = w.$$

2.4 Characterization

Lemma 2.1.
If $F(K,L)$ is HD1, then

$$F(K,L) = F_K(K,L)K + F_L(K,L)L$$

and $F_K(K,L)$, $F_L(K,L)$ are HD0.

2.4.1 Equalization of Capital-Labor Ratios

From firm's optimization in each sector

$$X\frac{r}{w} = \frac{F_K(K^i, XL^i)}{F_L(K^i, XL^i)} = \frac{F_K(K^i/(XL^i),1)}{F_L(K^i/(XL^i),1)}.$$

Since $X(t)\frac{r(t)}{w(t)}$ does not depend on i. there is some $k(t)$ s.t.

$$\frac{K^i(t)}{X(t)L^i(t)} = k(t) \quad \forall i.$$

2.4.2 Constant Relative Prices

From firm's FOCs and the previous result,

$$\frac{p^i}{p^j} = \frac{B^j}{B^i}.$$

In particular,

$$\frac{p^i}{p^M} = p^i = \frac{B^M}{B^i}.$$

- In CE prices determined by technology, not preferences (same growth, same relative price)

2.4.3 Consumer's Problem

Proposition 2.2.
From consumer's problem and results above, we can derive the following relations:

1. Intra-temporal

$$\frac{p^A(c^A-\gamma^A)}{\eta^A}=\frac{p^M c^M}{\eta^M}=\frac{p^S(c^S+\gamma^S)}{\eta^S}.$$

   Note that we can see from here that η_i controls the "adjusted" consumption share of each sector.

2. Equalized growth of adjusted consumption

$$\frac{\dot{c}^A}{c^A-\gamma^A}=\frac{\dot{c}^M}{c^M}=\frac{\dot{c}^S}{c^S+\gamma^S}.$$

3. Inter-temporal

$$\frac{\dot{c}}{c}=\frac{\dot{c}^M}{c^M}=\frac{1}{\theta}(r-\delta-\rho).$$

Proof.
Result 1: We can get the intra-temporal relations by dividing $[c^i]$ equations with each other.

Result 2: To see this, we take the following steps:

- Taking derivative of the intra-temporal relations with c^M with respect to t, we have

$$\begin{aligned} d\left(\frac{p^i(c^i-\gamma^i)}{\eta^i}\right)/dt =& d\left(\frac{c^M}{\eta^M}\right)/dt \\ \Rightarrow \quad \frac{\dot{p}^i(c^i-\gamma^i)+p^i\dot{c}^i}{\eta^i} =& \frac{\dot{c}^M}{\eta^M} \\ \Rightarrow \quad \dot{c}^i =& \frac{\eta^i\dot{c}^M}{p^i\eta^M} \end{aligned}$$

- Then we have

$$\frac{\dot{c}^i}{c^i-\gamma^i}=\frac{\eta^i\dot{c}^M}{p^i\eta^M}\frac{p^i\eta^M}{\eta^i c^M}=\frac{\dot{c}^M}{c^M},$$

  where we use the intra-temporal relations and the result from the previous step.

Result 3: To see this, we take the following steps:

- Taking derivative of $[c^M]$ with respect to t (recall $p^M = 1$) and plugging into $[K]$, we have

$$(1-\theta)\frac{\dot{c}}{c} - \frac{\dot{c}^M}{c^M} = -(r-\delta-\rho).$$

We would have the desired results if we have $\frac{\dot{c}}{c} = \frac{\dot{c}^M}{c^M}$.

- We take log of the definition of c and then take derivative with respect to t:

$$\begin{aligned}\frac{\dot{c}}{c} =& \eta^A \frac{\dot{c}^A}{c^A - \gamma^A} + \eta^M \frac{\dot{c}^M}{c^M} + \eta^S \frac{\dot{c}^S}{c^S - \gamma^S} \\ =& (\eta^A + \eta^M + \eta^S)\frac{\dot{c}^M}{c^M} \\ =& \frac{\dot{c}^M}{c^M}.\end{aligned}$$

□

2.4.4 Sectoral Reallocation and Structural Change

Result 2 above implies that

$$\frac{\dot{c}^A}{c^A} < \frac{\dot{c}^M}{c^M} < \frac{\dot{c}^S}{c^S}.$$

Since $c^A = Y^A$ and $c^S = Y^S$, we must have

$$\frac{\dot{Y}^A}{Y^A} < \frac{\dot{Y}^S}{Y^S}.$$

2.4.5 Kaldor Facts

We need to impose some assumptions on parameteres to hit the Kaldor facts.

Constant Growth Path (CGP): Equilibrium with constant interest rates.

- Necessary Conditions: Assume Constant Growth Path (CGP) exists and find what has to be true.
- Sufficient Conditions: Verify that the necessary conditions are also sufficient. ??

Definition 2.3. *Constant Growth Path in Acemoglu*
Constant Growth Path: In a CGP, the consumption aggregate grows at a constant rate, while output and employment in the three sectors grow at different rates. Given the preferences in (20.1), the constant growth rate of consumption also implies that the interest rate must be constant. (Acemoglu p.701)

Proposition 2.4. *"Knife-Edge" Condition for CGP*
In the above described economy, a CGP exists if and only if

$$\frac{\gamma^A}{B^A} = \frac{\gamma^S}{B^S}.$$

In a CGP, $k = k^*$ for all t, and moreover,

$$\frac{\dot{c}^A}{c^A} = g\frac{\dot{c}^A - \gamma^A}{c^A},\ \frac{\dot{c}^M}{c^M} = g,\ \frac{\dot{c}^S}{c^S} = g\frac{\dot{c}^S + \gamma^S}{c^S}.$$

Proof.
Necessity:

- If CGP exists, then from HH's intertemporal relation, we have $r = r(t)$. Recall our previous result that the capital-labor ratio is the same across sectors, we now have

$$\frac{K^i(t)}{XL^i(t)} = k.$$

  Rearranging and summing up across i, we have

$$K(t) = kX(t)\overline{L}.$$

  Thus, we have

$$\frac{\dot{K}}{K} = g.$$

- Rewriting the feasibility constraints with the CRS Lemma, we have

$$\begin{aligned} c^A =& B^A \left[F_K K^A + F_L X L^A\right] \\ c^M + gK =& B^M \left[F_K K^M + F_L X L^M\right] - \delta K \\ c^S =& B^S \left[F_K K^S + F_L X L^S\right] \end{aligned}$$

- Summing the feasibility constraints up, we now have

$$\begin{aligned} &\underbrace{p^A(c^A(t) - \gamma^A) + c^M(t) + p^S(c^S(t) + \gamma^S)}_{\text{grows at constant rate}} + \underbrace{[p^A\gamma^A - p^S\gamma^S]}_{\text{constant}} \\ &= \underbrace{B^M F(K(t), X(t)\overline{L} - (\delta + g)K(t))}_{\text{grows at rate } g} \end{aligned}$$

  We can only have constant growth here if

$$p^A\gamma^A - p^S\gamma^S = 0.$$

□

2.4.6 Labor Transition

Proposition 2.5. *Labor Transition*
We have

$$\frac{\dot{L}^A}{L^A} = -g\frac{\gamma^A/L^A}{B^A X F(k^*,1)},\ \frac{\dot{L}^M}{L^M} = 0,\ \frac{\dot{L}^S}{L^S} = g\frac{\gamma^S/L^S}{B^S X F(k^*,1)}.$$

Proof.
To see the relation for A and M, consider

- The feasibility for $i \in \{A, S\}$

  $$c^i = XL^iB^i(k^*,1)$$

  implies that

  $$\frac{\dot{c}^i}{c^i} = \frac{\dot{X}}{X} + \frac{\dot{L}^i}{L^i}.$$

- Use the relation from the previous proposition and the feasibility constraint, we have

  $$\begin{aligned}\frac{\dot{L}^A}{L^A} =& g\frac{c^A-\gamma^A}{c^A} - g \\ =& -g\frac{\gamma^A}{c^A} \\ =& -g\frac{\gamma^A/L^A}{B^A X F(k^*,1)} \\ \frac{\dot{L}^S}{L^S} =& g\frac{\gamma^S/L^S}{B^S X F(k^*,1)}\end{aligned}$$

To see the relation for M, consider

- The market clearing for labor is

  $$\overline{L} = L^A + L^M + L^S,$$

  which implies that

  $$\begin{aligned}0 =& \dot{L}^A + \dot{L}^M + \dot{L}^S \\ =& \dot{L}^M + g(-\frac{\gamma^A}{B^A} + \frac{\gamma^S}{B^S})\frac{1}{XF(k^*,1)} \\ =& \dot{L}^M.\end{aligned}$$

- Alternatively, we could take similar steps as those for A and M.

□

3 Structural Change: Supply Side

- Baumol's (1967) seminal work: "uneven growth" (non-balanced growth) will be a general feature of growth process because different sectors will grow at different rates owing to different rates of technological progress.
- Review some ideas based on Ngai and Pissarides (2007), who formalize Baumol's ideas.
- Rich patterns of structural change during early stages of development and those in more advanced economies today require models that combine supply-side and demand-side factors.
- Isolating these factors is both more tractable and also conceptually more transparent.

3.1 Setup

3.1.1 Household

Infinitely living representative household with exogenous labor supply $L = 1$.

Preferences:

$$U \equiv \int_{[0,\infty)} e^{-\rho t} \frac{c^{1-\theta} - 1}{1-\theta} dt,$$

where

$$c = \left(\sum_{i \in \{A,M,S\}} \eta^i (c^i)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

Budget constraint:

$$\sum_{i \in \{A,M,S\}} p^i c^i + \dot{K} = wL + (r - \delta)K.$$

3.1.2 Firm

Assume Cobb-Douglas production with constant but different productivity growth

$$Y^i = X^i \left(K^i\right)^\alpha \left(L^i\right)^{1-\alpha},$$

where

$$\frac{\dot{X}^i}{X^i} = g^i.$$

Firm's problem is

$$\max_{Y^i,K^i,L^i} p^i Y^i - wL^i - rK^i$$
$$\text{s.t. } Y^i = X^i \left(K^i\right)^\alpha \left(L^i\right)^{1-\alpha}.$$

Or equivalently,

$$\max_{K^i,L^i} p^i X^i \left(K^i\right)^\alpha \left(L^i\right)^{1-\alpha} - wL^i - rK^i.$$

3.1.3 Market Clearing

Labor and Capital

$$K^A + K^M + K^S = K, \quad L^A + L^M + L^S = L = 1.$$

Manufacturing good is used in production of investment good

$$I + c^M = Y^M, \quad \dot{K} = I - \delta K.$$

Agriculture and Service Goods

$$c^A = Y^A, \quad c^S = Y^S.$$

3.1.4 Competitive Equilibrium

Given initial K_0, collection of prices and quantities, such that

- consumers choose their quantities optimally given prices
- firms choose their quantities optimally given prices
- all markets clear

3.2 Optimality Conditions

3.2.1 Firm

Capital

$$p^i X^i \alpha \left(\frac{K^i}{L^i}\right)^{\alpha-1} = r.$$

Labor

$$p^i X^i (1-\alpha) \left(\frac{K^i}{L^i}\right)^{\alpha} = w.$$

3.2.2 Household

The FOCs for the household is

$$\begin{aligned} \frac{\partial H}{\partial c^i} &= c^{-\theta}\frac{\partial c}{\partial c^i} = \lambda p_i \\ -\frac{\partial H}{\partial K} &= -\lambda(r-\delta) = \dot{\lambda} - \rho\lambda, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial c}{\partial c^i} =& \left(\sum_i \eta^i (c^i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} \eta^i (c^i)^{-\frac{1}{\sigma}} \\ =& \eta^i \left(\frac{c^i}{c}\right)^{-\frac{1}{\sigma}} \end{aligned}$$

(This is the standard form for CES type aggregator.)

3.3 Characterization

3.3.1 Equalization of Capital-Labor Ratios

From firm's FOCs, we have

$$\frac{1-\alpha}{\alpha}\frac{K^i}{L^i} = \frac{w}{r}.$$

This implies that

$$k \equiv \frac{K^i}{L^i} = \frac{\alpha}{1-\alpha}\frac{w}{r}$$

is equalized across industries. We also know from this that $\frac{K}{L} = k$.

3.3.2 Relative Prices

From firm's FOCs, we have

$$\frac{p^i}{p^j} = \frac{X^j}{X^i}.$$

Therefore, relative prices fall in sectors with higher productivity growth.

3.3.3 Consumption Share

From consumer's FOCs, we have

$$\frac{c^i}{c^j} = \left(\frac{\eta^i}{\eta^j}\right)^{\sigma}\left(\frac{p^j}{p^i}\right)^{\sigma}$$

$$= \left(\frac{\eta^i}{\eta^j}\right)^{\sigma}\left(\frac{X^i}{X^j}\right)^{\sigma}$$
$$\frac{p^i c^i}{p^j c^j} = \left(\frac{\eta^i}{\eta^j}\right)^{\sigma}\left(\frac{X^i}{X^j}\right)^{\sigma-1}$$

This implies that the consumption share of good i

- is constant if $\sigma = 1$
- decreases for faster growing industry if $\sigma < 1$
- increases for faster growing industry if $\sigma > 1$

How to see σ as elasticity of substitution from here:

$$\ln\left(\frac{c^i}{c^j}\right) = -\sigma \ln\left(\frac{p^i}{p^j}\right) + \sigma \ln\left(\frac{\eta^i}{\eta^j}\right)$$
$$\Rightarrow \frac{\partial \ln(c^i/c^j)}{\partial \ln(p^i/p^j)} = -\sigma.$$

3.3.4 Relative Growth of Labor, Price, and Consumption

Proposition 3.1. *Growth Rates*
For $i, j \neq M$, we have the following relations.

$$\frac{\dot{L}^i}{L^i} - \frac{\dot{L}^j}{L^j} = (1-\sigma)(g^j - g^i)$$
$$\frac{\dot{p}^i}{p^i} - \frac{\dot{p}^j}{p^j} = (g^j - g^i)$$
$$\frac{\dot{c}^i}{c^i} - \frac{\dot{c}^j}{c^j} = -\sigma(g^j - g^i).$$

The relationship is more complicated for manufacturing goods because they are used for investments.

Proof.

- The first relation is from equalized captial-labor ratio and consumer's FOC.
  - From equal capital-labor ratio, we have
  $$c^i = X^i k^{\alpha} L^i.$$
  - Substituting this into consumer's FOCs, we have
  $$\frac{c^i}{c^j} = \frac{X^i k^{\alpha} L^i}{X^j k^{\alpha} L^j} = \left(\frac{\eta^i}{\eta^j}\right)^{\sigma}\left(\frac{X^i}{X^j}\right)^{\sigma}$$
  $$\Rightarrow \frac{L^i}{L^j} = \left(\frac{\eta^i}{\eta^j}\right)^{\sigma}\left(\frac{X^i}{X^j}\right)^{\sigma-1}$$

- The second relation is from firm's FOCs.

$$\frac{p^i}{p^j} = \frac{X^j}{X^i}.$$

- The third relation is from consumer's FOC.

□

Suppose demand is inelastic $\sigma < 1$, then
- prices of faster growing sector fall
- consumption share of that sector falls (even though physical consumption increases ??)
- labor outflows from that sector

Asymptotically, everyone works in the most stagnant sector.
- "Baumol's cost disease". But this is efficient allocation.

3.4 Uneven Growth and BGP

We define aggregate consumption C (note that this is different from consumption aggregator c) as

$$C = \sum_i p^i c^i.$$

Law of Motion: Summing up feasibility constraint (multiplied by corresponding p^i), we have

$$C + \dot{K} = X^M K^\alpha L^{1-\alpha} - \delta K.$$

Euler Equation: From household's FOCs, we have

$$\begin{aligned}\frac{\dot{C}}{C} &= (1-\theta)\frac{\dot{c}}{c} - \frac{\dot{\lambda}}{\lambda} \\ &= (1-\theta)\frac{\dot{c}}{c} + \frac{1}{\theta}(F_K(K,L) - \delta - \rho).\end{aligned}$$

Proof. Recall that household's FOCs are

$$c^{-\theta}\frac{\partial c}{\partial c^i} = \lambda p_i.$$

Summing the FOCs up and noticing that the CES aggregator c is CRS, we have

$$\begin{aligned}\sum_i \lambda p_i c^i &= \sum_i c^{-\theta}\frac{\partial c}{\partial c^i}c^i \\ \Rightarrow \quad \lambda C &= c^{1-\theta}.\end{aligned}$$

Therefore, we have

$$\frac{\dot{\lambda}}{\lambda} + \frac{\dot{C}}{C} = (1-\theta)\frac{\dot{c}}{c}.$$

□

If $\theta = 1$, we will be able to derive identical conditions to the neoclassical growth model.

- Ngai-Pissarides (2007) also show that q = 1 is a necessary condition.
- Nothing in what we said depended on 3 sector, same analysis applies to higher orders of disaggregation.

4 Structural Change: Demand vs Supply

4.1 Setup

Integrate both into preferences

$$u(c^A, c^M, c^S) = u\left(\left(\sum_i \omega_i^{\frac{1}{\sigma}} (c_i + \bar{c}_i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}\right),$$

with $\bar{c}^M = 0$.

- In the demand side story, we include the subsistence consumption. "Adjusted" consumption share is proportion to η_i. ($\sim$ non-homotheticity.)
- In the supply side story, we do not have the subsistence consumption. Consumption share is proportional to $(\eta_i)^\sigma$ as well as $(p_i)^{-\sigma}$.
- Here, we have subsistence consumption, and "adjusted" consumption share is also a function of p_i.

4.1.1 Expenditure Share

Let C be total consumption expenditure. Note its difference with pc, where c is the consumption aggregator and p is the ideal price index for CES demand, and we normalize all price with respect to it (i.e. $p = 1$).

Allocation of demand across sectors, conditional on C, solves the static problem

$$\begin{aligned} \max_{c_i} & \left(\sum_i \omega_i^{\frac{1}{\sigma}} (c_i + \bar{c}_i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ \text{s.t.} & \sum_i p_i c_i = C. \end{aligned}$$

Recall that p satisfies $\sum_i p_i(c_i + \bar{c}_i) = pc$.

The expenditure shares are given by

$$\begin{aligned} s_i \equiv \frac{p_i c_i}{C} &= \frac{p_i(\omega_i (p_i/p)^{-\sigma} c - \bar{c}_i)}{C} \\ &= \frac{p_i \omega_i (p_i/p)^{-\sigma} c}{Cpc/(pc)} - \frac{p_i c_i}{C} \\ &= \frac{p_i \omega_i (p_i/p)^{-\sigma} c \cdot pc}{C \sum_j p_j \omega_j (p_j/p)^{-\sigma} c} - \frac{p_i c_i}{C} \\ &= \frac{\omega_i (p_i)^{1-\sigma}}{\sum_j \omega_j (p_j)^{1-\sigma}} \frac{pc}{C} - \frac{p_i c_i}{C} \end{aligned}$$

$$= \frac{\omega_i (p_i)^{1-\sigma}}{\sum_j \omega_j (p_j)^{1-\sigma}} (1 + \sum_j \frac{p_j \bar{c}_j}{C}) - \frac{p_i c_i}{C}.$$

Note that we use the expression for CES demand

$$\hat{c}_i = c\omega_i \left(\frac{p_i}{p}\right)^{-\sigma}.$$

4.2 Empirical Evidence

By observing data on $p_{i,t}, c_{i,t}, C_t$, we can estimate all structural parameters using standard demand estimation methods.

There are different ways to conceptualize c_i.
1. Final Expenditure: purchases of food store $= A$, restaurant means $= S$.
2. Sectoral Value Added: VA of both types of food purchases created in A, M, S, and we can back them out using sectorial VA data.

4.2.1 Final Expenditure

Some observations
- Differential trends in prices/productivity

$$\frac{\dot{p}_S}{p_S} > \frac{\dot{p}_A}{p_A} > \frac{\dot{p}_M}{p_M}.$$

- Non-homotheticity must play a role
  - to simultaneously have c_A/c_A increasing when p_S/p_A also increases.
  - ?? Because in the supply side story where we have homotheticitiy, these two trends are in different direction?
- Both demand and supply stories are supported by the data
  - Demand: $\bar{c}_A < 0, \bar{c}_S > 0$
  - Supply: $\sigma < 1$

Horse-race
- Non-homotheticities/income effects are the driving force
- Technical progress (i.e. price growth rates) are very different across sectors, but it plays small role on reallocation with σ close to 1
- Even though $\bar{p}_A \bar{c}_A = \bar{p}_S \bar{c}_S$ condition does not hold, model fits the data well
  - Although BGP does not exist in a literal sense, equilibrium behaves approximately as such.

4.2.2 Sectorial Value Added

Data Construction
- GDP is constructed from sectorial value added

  - can classify A, M, S by those sectors
  - get sectorial price indices
- Subtract investments to obtain c_A, c_S, c_M using VA data
- Repeat the same demand estimation using these data

Observations
- Both demand and supply stories are supported by the data
- Supply-side story is now much more prominent
  - σ is quite close to 0: consumption across sectors is very poor substitute

4.2.3 Discussion

- Final expenditures and VA methods use the same data but different notions of what constitutes a "sector"
- Price trends had very different growth across sectors
  - competition: very different productivity growth rates
- How much that matters for structural change?
  - matters more if demand is less elastic
- There is much less substitutability in sectorial VA vs sectorial final expenditure
- Representation of demand via final expenditures is quite elastic
  - as restaurant prices increase, switch to buying food
  - in final expenditures shows up as substitution of c_A for c_S
- Representation of demand via VA is not elastic
  - both restaurants and grocery stores sell a bundle of $\{c_A, c_M, c_S\}$
  - low substitutability within a bundle (one orange and one cook required for one glass of orange juice no matter what the relative prices are)
- Bottom line: support in the data for both supply and demand stories
  - relative importance depends on what one means by "sector"

5 Trade and Structural Change

5.1 Background

East Asian Experience

- Japan 1960-1990
  - rapid growth rate of GDP driven by high productivity in manufacturing
  - share of manufacturing in GDP increased
- Obstfeld and Rogoff (1996): "Considering that Japan has had exceptionally high productivity growth in manufacturing relative to services, its experience is especially hard to square with productivity-based theories of manufacturing employment decline."

Trade is the missing piece

- One obviously missing ingredient in earlier model is trade
  - as you become more productive in sector i, your comparative advantage in sector i increases
  - can sell more of i stuff to the rest of the world
  - higher productivity in $i \Rightarrow$ higher demand from ROW
- A simple model of structural change with trade: Matsuyama (JEEA, 2009)

5.2 Setup

Two Countries: Home and Foreign (denoted with $*$)

- Each is endowed with one unit of the nontradeable factor (Labor).
- They differ only in Labor Productivity.

Three Goods:

- Agriculture, Numeraire (A); tradeable at zero cost;
  - No production. Endowment of y units.
- Manufacturing (M); tradeable at zero cost;
  - A unit of Home (Foreign) Labor produces X_M (X_M^*) units of M.
- Services (S): nontradeable;
  - A unit of Home (Foreign) Labor produces X_S (X_S^*) units of S.

5.2.1 Household

Utility Function

$$U = \begin{cases} (C_A - \gamma_A)^\alpha \left[\beta_M (C_M - \gamma)^\theta + \beta_S C_S^\theta\right]^{\frac{1-\alpha}{\theta}} & \theta \in (0,1) \\ (C_A - \gamma_A)^\alpha (C_M - \gamma)^{\beta_M(1-\alpha)} C_S^{\beta_S(1-\alpha)} & \theta = 0 \end{cases}.$$

Elasticity of substitution between M and S is

$$\sigma = \frac{1}{1-\theta}, \quad \text{i.e. } \theta = \frac{\sigma - 1}{\sigma}$$

so it is CES for these two goods.

Budget Constraint:

$$C_A + p_M C_M + p_S C_S \leq y + w.$$

5.2.2 Firm

Agriculture:

$$Y_A = y.$$

Manufacturing:

$$\max_{Y_M, L_M} p_M Y_M - w L_M \quad \text{s.t. } Y_M = X_M L_M.$$

Services:

$$\max_{Y_S, L_S} p_S Y_S - w L_S \quad \text{s.t. } Y_S = X_S L_S.$$

5.2.3 Market Clearing

Global Goods Feasibility

$$\begin{aligned} C_A + C_A^* &= 2y \\ C_M + C_M^* &= Y_M + Y_M^* \\ C_S &= Y_S \\ C_S^* &= Y_S^* \end{aligned}$$

Labor Feasibility:

$$\begin{aligned} L_M + L_S &= L = 1 \\ L_M^* + L_S^* &= L^* = 1 \end{aligned}$$

Free Trade in A and M

$$p_A = p_A^* = 1, \quad p_M = p_M^*.$$

5.3 Optimality Conditions

5.3.1 Household

Define

$$\hat{C}_A = C_A - \gamma_A, \ \hat{C}_M = C_M - \gamma, \ \hat{C}_S = C_S.$$

Notice that the utility function is Cobb-Douglas in C_A, and the C_M-C_S aggregator.

Rewriting the budget constraint, we have

$$\hat{C}_A + p_M \hat{C}_M + p_S \hat{C}_S \leq y + w - \gamma_A - p_M \gamma.$$

Then by making use of the demand function for Cobb-Douglas demand and CES demand, we have

$$\begin{aligned}
C_A &= \gamma_A + \alpha(y + w - \gamma_A - p_M \gamma), \\
C_M &= \gamma + \frac{\beta_M^\sigma p_M^{-\sigma}}{\beta_M^\sigma p_M^{1-\sigma} + \beta_S^\sigma p_S^{1-\sigma}}(1-\alpha)(y + w - \gamma_A - p_M \gamma), \\
C_S &= \frac{\beta_S^\sigma p_S^{-\sigma}}{\beta_M^\sigma p_M^{1-\sigma} + \beta_S^\sigma p_S^{1-\sigma}}(1-\alpha)(y + w - \gamma_A - p_M \gamma).
\end{aligned}$$

5.3.2 Firm

Firms are perfectly competitive.

Optimality in S

$$p_S = \frac{w}{X_S}, \quad p_S^* = \frac{w^*}{X_S^*}.$$

Optimality in M + free trade condition

$$p_M = \frac{w}{X_M} = \frac{w^*}{X_M^*}.$$

5.4 Characterization

5.4.1 Employment Share

From $C_A + C_A^* = 2y$, we can derive

$$y - \gamma_A = \frac{1}{1-\alpha} \frac{\alpha}{2} p_M (X_M + X_M^* + 2\gamma).$$

Substitute this into C_S, then we have

$$C_S = \frac{\frac{X_S}{X_M}\left(\frac{\alpha}{2}(X_M^* - X_M) + X_M - \gamma\right)}{1 + \left(\frac{\beta_M}{\beta_S}\right)^\sigma \left(\frac{X_S}{X_M}\right)^{1-\sigma}}$$

Then we have

$$L_M = 1 - L_S = 1 - \frac{C_S}{X_S}$$

$$= \frac{\frac{\alpha}{2}\left(1-\frac{X_M^*}{X_M}\right)+\frac{\gamma}{X_M}+\left(\frac{\beta_M}{\beta_S}\right)^{\sigma}\left(\frac{X_S}{X_M}\right)^{1-\sigma}}{1+\left(\frac{\beta_M}{\beta_S}\right)^{\sigma}\left(\frac{X_S}{X_M}\right)^{1-\sigma}}.$$

We need to restrict parameters so that $L_M \in (0,1)$ and $L_M^* \in (0,1)$.

5.4.2 Demand Side Effects

Focusing on demand-driven structural change: $\gamma > 0$ and $\sigma = 1$ (?? the latter emphasizes the non-homotheticity).

Then the employment share becomes

$$L_M = (1-\beta)\left[\frac{\alpha}{2}\left(1-\frac{X_M^*}{X_M}\right)+\frac{\gamma}{X_M}\right]+\beta,$$

where

$$\beta = \frac{\beta_M}{\beta_M+\beta_S}.$$

- The first term in the brackets is the "relative" effects, and the second term is the "absolute" effects.

Global Productivity Growth in M

$$\frac{\Delta X_M}{X_M} = \frac{\Delta X_M^*}{X_M^*} > 0.$$

Effect for labor is

$$\Delta L_M < 0, \quad \Delta L_M^* < 0.$$

National Productivity Growth in M

$$\frac{\Delta X_M}{X_M} > 0 = \frac{\Delta X_M^*}{X_M^*}.$$

Effect for labor is

$$sign[\Delta L_M] = sign[\frac{\alpha}{2}X_M^* - \gamma], \quad \Delta L_M^* < 0.$$

Trade Effect

- Ambiguity due to an additional force: trade effect
  - comparative advantage dictates that production of manufacturing goods is shifted to the country that is more efficient at producing that good.
  - whether home country experiences decline in manufacturing, depends on the relative strengths of non-homotheticity vs trade effects.
- Trade Effect can cause, in cross-section, a positive correlation between productivity gains and the employment share in M.

5.4.3 Supply Side Effects

Focusing on demand-driven structural change: $\gamma = 0$ and $\sigma < 1$.

Then the employment share becomes

$$L_M = \frac{\frac{\alpha}{2}\left(1 - \frac{X_M^*}{X_M}\right) + \left(\frac{\beta_M}{\beta_S}\right)^{\sigma}\left(\frac{X_S}{X_M}\right)^{1-\sigma}}{1 + \left(\frac{\beta_M}{\beta_S}\right)^{\sigma}\left(\frac{X_S}{X_M}\right)^{1-\sigma}}.$$

Global Productivity Growth in M

$$\frac{\Delta X_M}{X_M} = \frac{\Delta X_M^*}{X_M^*} > 0 = \frac{\Delta X_S}{X_S} = \frac{\Delta X_S^*}{X_S^*}.$$

Effect for labor is

$$\Delta L_M < 0, \quad \Delta L_M^* < 0.$$

National Productivity Growth in M

$$\frac{\Delta X_M}{X_M} > 0 = \frac{\Delta X_M^*}{X_M^*} = \frac{\Delta X_S}{X_S} = \frac{\Delta X_S^*}{X_S^*}.$$

Effect for labor is

$$sign[\Delta L_M] \text{ ambiguous}, \quad \Delta L_M^* < 0.$$

Ambiguity due to the two forces: Relative Supply & Trade Effects

5.4.4 Takeaways

- Higher productivity gains in Japanese M means that M must decline somewhere in the world, but not necessarily in Japan
- In cross-section of countries, M productivity can be positively correlated with M employment share, due to comparative advantage
- Global trend of M decline occurs due to productivity gains in M

6 Cross-Country Income Differences

6.1 Observations

Basic Observations

- Cross-country income differences persist over time.
- There is some evidence of convergence among OECD countries, but little evidence of convergence globally.

World and Neoclassical Growth

- All country on average grew with similar rates but had very different income levels
- Inconsistent with standard growth model where everyone can access the same technology (?? poorer countries are expected to grow faster?)

6.2 Some Explanations

Differences in Physical Capital

- Building (imputing) K from perpetual inventory method:

$$K_{t+1} = I_t + (1-\delta)K_t$$

- Impute K_0 as $I_0/(g+\delta)$ (steady state capital stock in Solow model).
- Conclusion: variation in physical capital explains no more than 20% of variation in output per capital
  - with possible exception of East Asian growth miracle

Differences in Human Capital

- "Quality-adjusted" L
- Starting point: Mincer regressions of returns to schooling S

$$\ln(w_i) = \mathbf{X}_i'\gamma + \phi S_i.$$

  People in rich countries go to school for longer.
- Assume ϕ do not vary much across poor and rich countries (Banerjee-Duflo (2005)), compute human capital
  - see Caselli (2005) handbook chapter on extensive discussion, robustness check
- Can explain at best 30% of income differences
  - If income differences is a factor of 10 or 20, this type of regressions will not bring you close to that.

Combining the Two

- Define $Y_{KH} = K^{\alpha}H^{1-\alpha}$, so that $Y = AY_{KH}$.
- How much of the cross-country variation does Y_{KH} explain?

$$\text{Var}\left[\ln Y\right] = \text{Var}\left[\ln Y_{KH}\right] + \text{Var}\left[\ln A\right] + 2\text{Cov}\left[\ln A, \ln Y_{KH}\right]$$

- If all countries had the same TFP

$$\mathrm{Var}\left[\ln A\right] = 0, \quad \mathrm{Cov}\left[\ln A, \ln Y_{KH}\right] = 0.$$

- So one measure of explanatory power of K and H is

$$\frac{\mathrm{Var}\left[\ln Y_{KH}\right]}{\mathrm{Var}\left[\ln Y\right]}.$$

- Y_{KH} can explain less than 40% of cross-country variation, even less of differences between top and bottom 10% of countries

TFP

- TFP accounts for most of cross-country dispersion of income?
- Why does it differ across countries? Next: looked at a deeper level
  - productivity differences across sectors
  - productivity differences within sectors

6.2.1 Productivity Differences Across Sectors

We shall look at agriculture and non-agriculture, as there is not enough internationally comparable data to have finer disaggregation.

Observations

- Poorer countries have higher employment shares of agriculture.
- Poorer countries have lower labor productivity in agriculture (measured as log agricultural GDP per worker)
- Poorer countries have lower labor productivity in non-agriculture, but the distance seems to be smaller

Conclusions and Counterfactuals

- Poor countries are mostly employed in agriculture and agriculture has particularly low productivity and that productivity is especially low in poor countries
- Consider 3 counterfactuals
  1. All countries have US level of agricultural GDP per worker but keep their employment shares and GDP per work in non-agriculture
  2. All countries have US level of non-agricultural GDP per worker but keep their employment shares and GDP per work in agriculture
  3. All countries have US labor shares, but keep their GDP per worker in both sectors

Counterfactuals Observations

- Elimination of productivity differences in agriculture (against US) could eliminate most of dispersion in income distribution
- Even adjustment of labor shares keeping GDP per worker the same could lead to substantial increase in income

7 Wedges

7.1 General Idea

Wedge decomposition is a powerful and popular diagnostic tool for understanding macroeconomic phenomena. It was originally developed by Chari, Kehoe, McGratten (Ecta, 2007) to study business cycles.

Cheremukhin, Golosov, Guriev, Tsyvinski "Industrialization and Economic Development of Russia through the lenses of the Neoclassical Growth Model" (ReStud, 2017)

- Motivation:
  - Structural transformation and reallocation of labor force from agriculture to manufacturing and services has been one of the central questions of growth and development
  - Even these days most poor countries are heavily agricultural. They also appear to be particular unproductive at that activity.
  - What slows reallocation to other sectors?
- What's Special
  - This paper uses neoclassical growth model to identify the sets of institutions and reforms that most affected structural transformation quantitatively
  - We develop a procedure that measures deviations of the predictions of the neoclassical growth model in quantities and prices ("wedge accounting")
- Wedge Accounting
  - Any set of policies (in market or command economic system) can be mapped into a system of taxes ("wedges") in a neoclassical growth model
  - Different policies map into different wedges
  - Consistent procedure to measure importance of different policies across different economic regimes/institutions
  - Broader methodological applicability beyond analysis of Russia.

7.2 Historical Background on Russia

Economic Policies in Russia Pre-1913

- Serfdom abolished in 1861
- Land belongs to communal property
  - land rents shared equally
  - many communes do not allow sale of individual land rights
- Small industrial sector

- tsar is historically afraid of political challenge from bourgeoisie, keeps large barriers
- hard to set up companies, run them without state interference
- prevalence of cartels and monopolies

- Large country with limited transportation network
  - regional markets not well integrated
  - many peasants have little participation in market activity

Economic Policies in Russia Post-1928

- Industrialization: company managers are encouraged to massively expand industrial production

- Collectivization: collective farms are created in countryside, by 1935 most peasants are employed there
  - results in famines in villages in 1932-33
  - passport system introduced to stem flow of peasants to the cities

- Rationing of consumer goods in 1929-34
  - by 1935 free markets at which farmers sell agricultural goods, buy manufacturing
  - state store prices equalize with those prices by 1937

- Politburo sets general quantities targets and some prices; enterprise-level quantities and prices emerge from decentralized contracting between state ministries and individual enterprises

7.3 Setup

7.3.1 Household

Preferences

$$\sum_{t=1}^{\infty} \beta^t \frac{U(c_t^A, c_t^M)^{1-\rho} - 1}{1-\rho},$$

where

$$U(c^A, c^M) = \left[\sum_{i \in \{A,M\}} \eta_i^{\frac{1}{\sigma}} \left(c^i - \gamma_i \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}.$$

Each household inelastically supply 1 unit of labor. Population growth is exogenous and there are N_t households at time t.[1]

[1] This follows more closely with the paper than the slide.

7.3.2 Firm

Firm's Problem ($i \in \{A, M\}$)

$$\max_{Y^i, K^i, L^i} p^i Y^i - rK^i - wN^i$$
$$\text{s.t. } Y_t^i = X_t^i (K_t^i)^{\alpha_i} (N_t^i)^{1-\alpha_i}.$$

7.3.3 Market Clearing

Labor

$$N_t^A + N_t^M = N_t.$$

Capital

$$K_t^A + K_t^M = K_t.$$

Capital Law of Motion

$$K_{t+1} = (1-\delta)K_t + I_t.$$

Goods

$$C_t^M + I_t = Y_t^M, \quad C_t^A = Y_t^A,$$

where

$$C_t^i = N_t c_t^i.$$

Note that we normalize prices with respect to that of M so that the law of motion and market clearing conditions make sense (or are easier to write).

7.4 Optimal Allocation and Wedges

7.4.1 Household

We can write household's Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\beta^t N_t \frac{U_t^{1-\rho} - 1}{1-\rho} + \lambda_t \left(r_t K_t + w_t N_t - \sum_i p^i C^i - (K_{t+1} - (1-\delta)K_t) \right) \right).$$

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t^i} = \beta^t N_t U_t^{-\rho} U_{i,t} - \lambda_t p_t^i = 0,$$
$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t + \lambda_{t+1}(1 + r_{t+1} - \delta) = 0.$$

7.4.2 Firm

The firm's standard FOCs are

$$F^i_{K,t}(K^i_t, N^i_t) = r_t, \quad F^i_{N,t}(K^i_t, N^i_t) = w_t.$$

7.4.3 Optimality Conditions

Intra-Temporal

From household's FOCs, we have

$$\frac{\partial \mathcal{L}}{\partial c^M_t} \Bigg/ \frac{\partial \mathcal{L}}{\partial c^A_t} \;\Rightarrow\; \frac{U_{M,t}}{U_{A,t}} = \frac{p^M_t}{p^A_t}.$$

From firm's FOCs, we have

$$\frac{p^M_t F^M_{N,t}}{p^A_t F^A_{N,t}} = 1, \quad \frac{p^M_t F^M_{K,t}}{p^A_t F^A_{K,t}} = 1.$$

Combining these results, we have

$$1 = \frac{U_{M,t} F^M_{N,t}}{U_{A,t} F^A_{N,t}}, \quad 1 = \frac{U_{M,t} F^M_{K,t}}{U_{A,t} F^A_{K,t}}.$$

Note that the resulting equations do not depend on prices.

We can decompose the equations with prices as

$$\begin{aligned} 1 &= \frac{U_{M,t} F^M_{N,t}}{U_{A,t} F^A_{N,t}} = \frac{U_{M,t}/p^M_t}{U_{A,t}/p^A_t} \times \frac{p^M_t F^M_{N,t}/w_{M,t}}{p^A_t F^A_{N,t}/w_{A,t}} \times \frac{w_{M,t}}{w_{A,t}}. \\ 1 &= \frac{U_{M,t} F^M_{K,t}}{U_{A,t} F^A_{K,t}} = \frac{U_{M,t}/p^M_t}{U_{A,t}/p^A_t} \times \frac{p^M_t F^M_{K,t}/r_{M,t}}{p^A_t F^A_{K,t}/r_{A,t}} \times \frac{r_{M,t}}{r_{A,t}}. \end{aligned}$$

Note that in the competitive equilibrium, each of the three components equal to 1.

Inter-Temporal

From household's FOCs, we have

$$\begin{aligned} & 1 = \frac{\lambda_{t+1}}{\lambda_t}(1 + r_{t+1} - \delta) \\ \Rightarrow\; & 1 = \frac{\beta^{t+1} N_{t+1} U^{-\rho}_{t+1} U_{M,t+1}/p^M_{t+1}}{\beta^t N_t U^{-\rho}_t U_{M,t}/p^M_t}(1 + r_{t+1} - \delta) \\ \Rightarrow\; & 1 = \frac{\beta N_{t+1} U^{-\rho}_{t+1} U_{M,t+1}}{N_t U^{-\rho}_t U_{M,t}}(1 + r_{t+1} - \delta) \end{aligned}$$

Note that we used the fact that we normalize prices with respect to p^M.

Substituting r_t by using firm's FOCs, we have

$$1 = \frac{\beta N_{t+1} U_{t+1}^{-\rho} U_{M,t+1}}{N_t U_t^{-\rho} U_{M,t}} (1 + F_{K,t+1}^M - \delta).$$

Golosov has confirmed that he used $U_{i,t}$ to denote the derivative of $N_t U^{1-\rho}$ with respect to i, so our expression here would look slightly different from his but are substantially the same.

7.4.4 Definition of Wedges

Taking any arbitrary collection of allocation and prices, we define wedges as follows. Note that this arbitrary collection of allocation and prices can be supported as a CE with taxes, where taxes are set to wedges and their components. (Any collection of policies/distortions is equivalent to some combination of taxes.)

Intersectoral Labor Wedge $\tau_{w,t}$

$$1 + \tau_{w,t} = \frac{U_{M,t} F_{N,t}^M}{U_{A,t} F_{N,t}^A} = \underbrace{\frac{U_{M,t}/p_t^M}{U_{A,t}/p_t^A}}_{\text{Consumption Component}} \times \underbrace{\frac{p_t^M F_{N,t}^M / w_{M,t}}{p_t^A F_{N,t}^A / w_{A,t}}}_{\text{Production Component}} \times \underbrace{\frac{w_{M,t}}{w_{A,t}}}_{\text{Mobility Component}}.$$

Intersectoral Capital Wedge $\tau_{R,t}$

$$1 + \tau_{r,t} = \frac{U_{M,t} F_{K,t}^M}{U_{A,t} F_{K,t}^A} = \underbrace{\frac{U_{M,t}/p_t^M}{U_{A,t}/p_t^A}}_{\text{Consumption Component}} \times \underbrace{\frac{p_t^M F_{K,t}^M / r_{M,t}}{p_t^A F_{K,t}^A / r_{A,t}}}_{\text{Production Component}} \times \underbrace{\frac{r_{M,t}}{r_{A,t}}}_{\text{Mobility Component}}.$$

Intertemporal Capital Wedge $\tau_{K,t}$

$$1 + \tau_{K,t} = \frac{\beta N_{t+1} U_{M,t+1}}{N_t U_{M,t}} (1 + F_{K,t+1}^M - \delta).$$

7.5 Wedge Accounting

Key Observations

- Any economic policy is equivalent to a set of taxes and transfers in a standard competitive equilibrium
  - the studied economy does not need to be competitive, or use market mechanisms for resource allocations, etc
- Different policies will map into different combination of wedge

  - measure wedges and their quantitative importance ⇒ see which policies matter and how much
- Several policies can map into the same wedge
  - cannot pin down specific policy, but can distinguish between broad classes of explanations
- This approach does not require us to assume that our assumption on preferences and technologies are correct
  - if preferences/technologies are different ⇒ shows up as wedges in this analysis
  - different theories about preferences/technologies have different implications about such wedges
- If you know preferences and technology: can measure total distortions or wedges (aka "taxes") in the data

7.6 Policies to Wedges

Mapping of Tsarist Frictions to Wedges

- Obschina: land pre 1910 is in a communal property of a village. If peasant leave a village, he loses land rent

$$\text{mobility component} > 1$$

- Limited competition: large prevalence of monopolies and cartels in manufacturing, barriers to setting corporations

$$\text{production component} > 1$$

- Limited market participation: a lot of peasants poorly integrated in market economy, mostly produce for own consumption

$$\text{consumption component} > 1$$

- Costly human capital acquisition:

$$\text{mobility component} > 1$$

Mapping of Soviet Policies to Wedges

- Rationing/non-market clearing prices:

$$\text{consumption components} \quad \uparrow, \downarrow$$

- Big push: a common story of success of industrialization (e.g. Murphy-Shleifer-Vishny)

$$\begin{aligned} &\text{TFP in manufacturing} \quad \uparrow \\ &\text{production component} \quad \uparrow \end{aligned}$$

- Expansion of industry/creating of collective farms (monopsonist on ag labor market):

$$\text{production component} \quad \downarrow$$

7.7 Empirical Results

7.7.1 Strategy

- Take data from historican sources
- Use standard model parameters for labor shares and preferences
  - Caselli and Coleman, Hayashi-Prescott, Buera-Kaboski, Herrendorf-Rogerson-Valentinyi
- Compute

$$\left\{X^M(t), X^A(t), \tau^W(t), \tau^R(t), \tau^K(t), \text{components}(t)\right\}_t .$$

7.7.2 Discussion and Big Picture

Discussion

- Tsarist Russia had very high wedges prior to 1914, preventing reallocation
  - production component of intratemporal wedges particularly high
- All wedges in 1932-1940 become worse, except production components
  - by 1940, this wedge become 1
- Can decompose further into contribution of numerator and denominator

$$\begin{aligned}&\Delta \ln(\text{production component})\\=&\Delta \ln(\text{mark-up in non-ag}) - \Delta \ln(\text{mark-up in ag})\end{aligned}$$

- 88% of drop in production component comes from decrease in non-ag mark ups

Big Picture

- Small labor share in non-agriculture in tsarist Russia driven by monopoly distortions
- Removal of monopoly distortions can lead to structural transformation
  - can be done by removal of barriers to entry
  - or by incentivizing managers to increase production using threat of prosecution
- Soviet approach reduced the monopoly distortions but also lowered productivity
  - also led to mass famine and political terror

8 Misallocation within Sectors

Most of the differences in GDP is driven by differences in TFP, even on sectoral level. We will try to understand productivity differences within sectors.

Why monopolistic competition?

- Perfect competition: all firms set price at marginal cost, most competitive firm gets the whole market
  - Industry TFP = TFP of most effcient firm
  - Not very realistic
- Monopolistic competition: firms have some monopoly power, charge a mark up over marginal cost
  - Inefficient firms operate in equilibrium
- Simplest monopolistic competition: all charge the same markup

8.1 Setup

Simplest Model

- 1 intermediate sector, 1 final sector
- Intermediate Sector: Firm i produces differentiated product Y_i with technology

$$Y_i = A_i K_i^{\alpha} L_i^{1-\alpha}$$

  - Firm i is monopolist for good i and charges price P_i.
  - Note that factor share is the same within sector.
- Final Sector: Competitive.

8.1.1 Final Sector

The final firm solves

$$\max_{Y_i} PY - \int P_i Y_i di$$
$$\text{s.t. } Y = \left(\int Y_i^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}.$$

Note that we normalize with respect to the ideal price index of CES demand so that $P = 1$. The ideal price index is given by

$$P = \left(\int P_i^{1-\sigma} di \right)^{\frac{1}{1-\sigma}}$$

The FOC yields the demand for good i as

$$Y_i = Y \left(\frac{P_i}{P}\right)^{-\sigma}.$$

Note that this implies downward demand curve for good i with constant elasticity σ.

- We assume $\sigma > 1$ throughout, or no equilibrium exists.

8.1.2 Intermediate Sector (Undistorted)

Firm has monopoly power and faces the problem

$$\begin{aligned} \max_{P_i, Y_i, L_i, K_i} \; & P_i Y_i - wL_i - rK_i \\ \text{s.t.} \; & Y_i = Y \left(\frac{P_i}{P}\right)^{-\sigma} \\ & Y_i = A_i K_i^{\alpha} L_i^{1-\alpha}. \end{aligned}$$

Working through the algebra, we have

$$P_i = \frac{\sigma}{\sigma - 1} \frac{1}{A_i} \left(\frac{r}{\alpha}\right)^{\alpha} \left(\frac{w}{1-\alpha}\right)^{1-\alpha}.$$

Next, we will derive the marginal cost and show that this expression implies constant markup.

Proof. The Lagrangian is

$$\mathcal{L} = P_i Y_i - wL_i - rK_i + \lambda_i \left(Y \left(\frac{P_i}{P}\right)^{-\sigma} - Y_i \right) + \mu_i (A_i K_i^{\alpha} L_i^{1-\alpha} - Y_i).$$

The FOCs are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial P_i} &= Y_i - \lambda_i Y \sigma \left(\frac{P_i}{P}\right)^{-\sigma-1} \frac{1}{P} = 0 && [P_i] \\ \frac{\partial \mathcal{L}}{\partial Y_i} &= P_i - \lambda_i - \mu_i = 0 && [Y_i] \\ \frac{\partial \mathcal{L}}{\partial K_i} &= -r + \mu_i A_i \alpha K_i^{\alpha-1} L_i^{1-\alpha} = 0 && [K_i] \\ \frac{\partial \mathcal{L}}{\partial L_i} &= -w + \mu_i A_i K_i^{\alpha} (1-\alpha) L_i^{-\alpha} = 0 && [L_i] \\ \frac{\partial \mathcal{L}}{\partial \lambda_i} &= Y \left(\frac{P_i}{P}\right)^{-\sigma} - Y_i = 0 && [\lambda_i] \\ \frac{\partial \mathcal{L}}{\partial \mu_i} &= A_i K_i^{\alpha} L_i^{1-\alpha} - Y_i = 0 && [\mu_i] \end{aligned}$$

- $[L_i]/[K_i]$:

$$\frac{1-\alpha}{\alpha}\frac{K}{L} = \frac{w}{r}$$

Then we have

$$\begin{aligned}\mu_i =& \frac{w}{1-\alpha}\frac{1}{A_i}\left(\frac{K_i}{L_i}\right)^{-\alpha}\\ =& \frac{1}{A_i}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}.\end{aligned}$$

- $[P_i]$ & λ_i:

$$\begin{aligned}\lambda_i =& Y_i \Big/ \left[Y\sigma\left(\frac{P_i}{P}\right)^{-\sigma-1}\frac{1}{P}\right]\\ =& \frac{1}{\sigma}P_i.\end{aligned}$$

- $[Y_i]$:

$$\begin{aligned}0 =& P_i - \frac{1}{\sigma}P_i - A_i\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\\ \Rightarrow \quad P_i =& \frac{\sigma}{\sigma-1}\frac{1}{A_i}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}.\end{aligned}$$

□

Marginal Cost

Firm faces the cost minimization problem

$$\begin{aligned}C(Y_i) = \min_{K_i,L_i} wL_i + rK_i\\ \text{s.t. } Y_i = A_i K_i^{\alpha} L_i^{1-\alpha}.\end{aligned}$$

FOCs imply

$$\frac{K_i}{L_i} = \frac{\alpha}{1-\alpha}\frac{w}{r}, \quad \lambda_i = \frac{1}{A_i}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}.$$

Marginal cost of firm i is, via the Envelope theorem,

$$C'(Y_i) = \lambda_i = \frac{1}{A_i}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}.$$

Constant Markup

Recall the optimal price

$$P_i = \underbrace{\frac{\sigma}{\sigma-1}}_{\text{Markup}>1} \underbrace{\frac{1}{A_i}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}}_{\text{Marginal Cost}}.$$

All firms charge constant mark up over marginal costs. We also have perfect competition as the limit $\sigma \to \infty$.

K_i and L_i

We can solve for K_i and L_i as

$$K_i = A_i^{\sigma-1} Y P^{\sigma} \left(\frac{\sigma}{\sigma-1}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\right)^{-\sigma}\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{1-\alpha}$$
$$L_i = A_i^{\sigma-1} Y P^{\sigma} \left(\frac{\sigma}{\sigma-1}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\right)^{-\sigma}\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{-\alpha}.$$

Proof. Combining $[\lambda_i]$ and $[\mu_i]$, and then using expressions for $\frac{K_i}{L_i}$ and P_i, we have

$$\begin{aligned} & Y\left(\frac{P_i}{P}\right)^{-\sigma} = A_i K_i^{\alpha} L_i^{1-\alpha} \\ \Rightarrow\ & Y\left(\frac{P_i}{P}\right)^{-\sigma} = A_i\left(\frac{K_i}{L_i}\right)^{\alpha} L_i \\ \Rightarrow\ & L_i = A_i^{-1} Y P^{\sigma} P_i^{-\sigma}\left(\frac{K_i}{L_i}\right)^{-\alpha} \\ \Rightarrow\ & L_i = A_i^{\sigma-1} Y P^{\sigma}\left(\frac{\sigma}{\sigma-1}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\right)^{-\sigma}\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{-\alpha}. \end{aligned}$$

Similar derivation follows for K_i. □

These implies that we have

$$K_i, L_i \propto A_i^{\sigma-1}.$$

8.1.3 Intermediate Sector (Distorted)

Suppose allocations are distorted with firm-specific wedges $\tau_{Y,i}, \tau_{K,i}$ (note that we essentially normalize labor wedges to 1).

Firm faces the problem

$$\max_{P_i,Y_i,L_i,K_i} (1-\tau_{Y,i})P_iY_i - wL_i - (1+\tau_{K,i})rK_i$$
$$\text{s.t. } Y_i = Y\left(\frac{P_i}{P}\right)^{-\sigma}$$
$$Y_i = A_iK_i^{\alpha}L_i^{1-\alpha}.$$

Solution to the problem is given by

$$\frac{K_i}{L_i} = \frac{\alpha}{1-\alpha}\frac{w}{r}\frac{1}{1+\tau_{K,i}}, \; MPRL_i = \frac{w}{1-\tau_{Y,i}}, \; MPRK_i = r\frac{1+\tau_{K,i}}{1-\tau_{Y,i}}.$$

Proof. Following the same steps as above, we have

$$P_i = \frac{(1-\tau_{K,i})^{\alpha}}{1-\tau_{Y,i}}\frac{\sigma}{\sigma-1}\frac{1}{A_i}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}.$$

We also have

$$K_i \propto A_i^{\sigma-1}\left(\frac{(1-\tau_{K,i})^{\alpha}}{1-\tau_{Y,i}}\right)^{-\sigma}\left(\frac{1}{1-\tau_{K,i}}\right)^{-\alpha},$$
$$L_i \propto A_i^{\sigma-1}\left(\frac{(1-\tau_{K,i})^{\alpha}}{1-\tau_{Y,i}}\right)^{-\sigma}\left(\frac{1}{1-\tau_{K,i}}\right)^{1-\alpha}.$$

□

8.2 Measures of Efficiency

8.2.1 $TFPQ_i$ and $TFPR_i$ (Undistorted)

Physical Productivity

$$TFPQ_i := \frac{Y_i}{K_i^{\alpha}L_i^{1-\alpha}} = A_i.$$

Revenue Productivity

$$TFPR_i := \frac{P_iY_i}{K_i^{\alpha}L_i^{1-\alpha}} = P_iA_i.$$

Undistorted firm optimization implies

$$TFPR_i = TFPR_j \;\; \forall i,j.$$

8.2.2 Industry TFP (Undistorted)

Industry Value Added

$$PY = \int P_i Y_i di.$$

Industry Capital Stock and Employment

$$K = \int K_i di, \quad L = \int L_i di.$$

Therefore, in the data TFP will show up as

$$Y = TFP \times K^\alpha \times L^{1-\alpha}.$$

The production function of the final sector

$$\begin{aligned} Y &= \left(\int Y_i^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\int \left[A_i \left(\frac{K_i}{K}\right)^\alpha \left(\frac{L_i}{L}\right)^{1-\alpha}\right]^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} K^\alpha L^{1-\alpha}. \end{aligned}$$

(where we simply use the definition of technology Y_i) gives us the definition of TFP as

$$TFP^{undist} := \left(\int \left[A_i \left(\frac{K_i}{K}\right)^\alpha \left(\frac{L_i}{L}\right)^{1-\alpha}\right]^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}.$$

The expression of TFP can be then simplied as

$$\begin{aligned} TFP^{undist} &= \left(\int \left[A_i \left(\frac{K_i}{K}\right)^\alpha \left(\frac{L_i}{L}\right)^{1-\alpha}\right]^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\int \left[A_i \frac{A_i^{\sigma-1}}{\int A_j^{\sigma-1} dj}\right]^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\frac{\int A_i^{\sigma-1} di}{(\int A_i^{\sigma-1} di)^{\frac{\sigma-1}{\sigma}}}\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\left[\int A_i^{\sigma-1} di\right]^{1-\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\int A_i^{\sigma-1} di\right)^{\frac{1}{\sigma-1}}. \end{aligned}$$

- In the second equality, we use $K_i, L_i \propto A_i^{\sigma-1}$ and $K = \int K_i di, L = \int L_i di$.

8.2.3 Simplifying TFP (Undistorted)

We want to simply

$$\ln(TFP^{undist}) = \frac{1}{\sigma - 1} \ln\left(\int A_i^{\sigma-1} di\right).$$

WLOG, we can write

$$\ln(A_i) = \overline{a} + a_i,$$

where

$$\overline{a} = \int \ln(A_i) di \quad \text{and} \quad a_i = \ln(A_i) - \overline{a}.$$

This definition implies that $\int a_i di = 0$. Note that we shall denote integral with expectations.

For any $x \geq 0$, define

$$F(x) = \frac{1}{\sigma - 1} \ln\left(\mathrm{E}\left[e^{(\sigma-1)(\overline{a}+xa_i)}\right]\right).$$

Note that $F(1) = \ln(TFP^{undist})$.

Using standard Taylor expansion, we have

$$F(1) \approx F(0) + F'(0) + \frac{1}{2}F''(0),$$

which is

$$\ln(TFP^{undist}) \approx \mathrm{E}\left[\ln(A_i)\right] + \frac{\sigma - 1}{2}\mathrm{Var}\left[\ln(A_i)\right].$$

Proof. To see this, we can calculate $F'(x)$ as

$$\begin{aligned} F'(x) &= \frac{1}{\sigma - 1} \frac{\mathrm{E}\left[e^{(\sigma-1)(\overline{a}+xa_i)}(\sigma - 1)a_i\right]}{\mathrm{E}\left[e^{(\sigma-1)(\overline{a}+xa_i)}\right]} \\ \Rightarrow \quad F'(0) &= 0, \end{aligned}$$

where we assume that we can exchange the order of integration and differentiation. □

8.2.4 Misallocation and TFP

Now $TFPR_i$ are no longer equalized for all i, and we have

$$TFPR_i \propto \frac{(1+\tau_{K,i})^{\alpha}}{1-\tau_{Y,i}}.$$

The sectoral TFP is given by

$$TFP^{dist} = \left(\int \left(A_i \frac{\overline{TFPR}}{TFPR_i} \right)^{\sigma-1} di \right)^{\frac{1}{\sigma-1}},$$

where

$$\frac{1}{\overline{TFPR}} = \int \frac{1}{TFPR_i} \frac{P_i Y_i}{PY} di.$$

This is a geometric average.

If $\ln(A_i)$ and $\ln(TFPR_i)$ are uncorrelated, we get

$$\ln(TFP^{dist}) \approx \ln(TFP^{undist}) - \frac{\sigma}{2} \mathrm{Var}\left[\ln(TFPR_i)\right].$$

Proof.

- $TFPR_i$: This follows directly from the expression of P_i with distortion and the definition of $TFPR_i$.
- TFP^{dist}

□

8.3 Empirical

In firm census data, we observe

$$L_i^j,\ K_i^j,\ \mathcal{R}_i^j = P_i^j Y_i^j,\ \mathcal{L}_i^j = w^j L_i^j.$$

8.3.1 $TFPR_i^j$

Then we can get $TFPR_i^j$ directly as

$$TFPR_i^j = \frac{\mathcal{R}_i^j}{(K_i^j)^{\alpha}(L_i^j)^{1-\alpha}}.$$

We can also get wedges from firm optimization as

$$1+\tau_{K,i}^j = \frac{\alpha}{1-\alpha} \frac{\mathcal{L}_i^j}{rK_i^j}$$

$$1-\tau_{Y,i}^j = \frac{\sigma}{\sigma-1} \frac{\mathcal{L}_i^j}{(1-\alpha)\mathcal{R}_i^j}.$$

8.3.2 $TFPQ_i^j$

Firm level productivity is

$$TFPQ_i^j = \frac{Y_i^j}{(K_i^j)^\alpha (L_i^j)^{1-\alpha}},$$

but we do not observe Y_i^j.

From $Y_i^j = Y^j (P_i^j/P)^{-\sigma}$, we get

$$Y_i^j = (Y^j)^{-\frac{1}{1-\sigma}} \left(P_i^j Y_i^j\right)^{\frac{\sigma}{\sigma-1}} = const^j (\mathcal{R}_i^j)^{\frac{\sigma}{\sigma-1}}.$$

Therefore,

$$TFPQ_i^j = const^j \times \frac{(\mathcal{R}_i^j)^{\frac{\sigma}{\sigma-1}}}{(K_i^j)^\alpha (L_i^j)^{1-\alpha}}.$$

8.3.3 Output Loss due to Misallocation

We have

$$\begin{aligned}
\frac{Y^{dist}}{Y^{undist}} &= \frac{TFP^{dist} \times K^\alpha L^{1-\alpha}}{TFP^{undist} \times K^\alpha L^{1-\alpha}} \\
&= \left(\int \left(\frac{A_i}{\overline{A}^{undist}} \frac{\overline{TFPR}}{TFPR_i} \right)^{\sigma-1} di \right)^{\frac{1}{\sigma-1}},
\end{aligned}$$

where

$$\overline{A}^{undist} = \left(\int A_i^{\sigma-1} di \right)^{\frac{1}{\sigma-1}}.$$

9 Static Monopolistic Competition

The CES consumption aggregator C is given by

$$C = \left(\int_{[0,1]} C(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

The ideal price index P for the CES aggregator is given by

$$P = \left(\int_{[0,1]} (P(i))^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$$

Note that P ensures that

$$\frac{P(i)}{P} = \left(\frac{C(i)}{C} \right)^{-\frac{1}{\epsilon}}, \quad PC = \int_{[0,1]} P(i)C(i)di.$$

9.1 Social Planner's Problem

The social planner's problem is

$$\begin{aligned}
&\max_{C,N} \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi} \\
\text{s.t. } \; & C(i) = AN(i) && \text{[Technology]} \\
& N = \int N(i)di. && \text{[Feasibility]}
\end{aligned}$$

9.1.1 Solution

Proposition 9.1. *Equalized Consumption*
Given N, C is maximized when $C(i)$ is equalized.

Proof. The Lagrangian is

$$\mathcal{L} = \left(\int_{[0,1]} (AN(i))^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} + \lambda \left(N - \int N(i)di \right).$$

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial N(i)} = C^{\frac{1}{\epsilon}} A(AN(i))^{-\frac{1}{\epsilon}} - \lambda = 0.$$

This implies that all $N(i)$ and thus $C(i)$ is equalized.

Note that we may also see this intuitively from the convexity of the aggregator. (??) □

Therefore, we can write the problem as

$$\begin{aligned} &\max_{C,N} \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi} \\ &\text{s.t.} \;\; C = AN. \end{aligned}$$

The FOCs are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C} &= C^{1-\sigma} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial N} &= -N^{\varphi} + \lambda A = 0. \end{aligned}$$

The solution can be characterized by the system of equations

$$\begin{aligned} N^{\varphi} C^{\sigma} &= A \\ C &= AN. \end{aligned}$$

9.2 Competitive Equilibrium

9.2.1 Household

The household's problem is

$$\begin{aligned} &\max_{C,N} \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi} \\ \text{s.t.} \quad &\int P(i)C(i)di = (1-\tau)WN + D + T. \end{aligned}$$

The household's Lagrangian is

$$\mathcal{L} = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi} + \lambda\left((1-\tau)WN + D + T - \int P(i)C(i)di\right).$$

The FOCs are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C(i)} &= C^{-\sigma} C^{\frac{1}{\epsilon}} C(i)^{-\frac{1}{\epsilon}} - \lambda P(i) = 0 && [C(i)] \\ \frac{\partial \mathcal{L}}{\partial N} &= -N^{\varphi} + \lambda(1-\tau)W = 0 && [N] \end{aligned}$$

The ideal price index satisfies the relation

$$\frac{P(i)}{P} = \left(\frac{C(i)}{C}\right)^{-\frac{1}{\epsilon}}.$$

Rearranging this relation and use it to rewrite $C(i)$, we have

$$C(i) = \left(\frac{P(i)}{P}\right)^{-\epsilon} C.$$

- The derivation of $C(i)$ with respect to $P(i)$, which we often use, is

$$\begin{aligned}\frac{\partial C(i)}{\partial P(i)} &= -\epsilon\left(\frac{P(i)}{P}\right)^{-\epsilon-1}\frac{1}{P}C \\ &= -\epsilon\frac{C(i)}{P(i)}.\end{aligned}$$

Then $[C(i)]$ can be written as

$$C^{-\sigma} = \lambda P.$$

Combining with $[N]$, we have

$$N^{\varphi}C^{\sigma} = (1-\tau)\frac{W}{P}.$$

This allows us to rewrite the household's problem with two-stage budgeting, where the first stage is optimization of the aggregate variable, and the second stage is optimization of the consumption bundle:

$$\begin{aligned}&\max_{C,N}\frac{C^{1-\sigma}}{1-\sigma}-\frac{N^{1+\varphi}}{1+\varphi}, \\ \text{s.t.}\quad & PC = (1-\tau)WN + D + T.\end{aligned}$$

9.2.2 Firm

The firm's problem is

$$\begin{aligned}\Pi = &\max_{P,Y,N} P(i)Y(i) - WN(i) \\ \text{s.t.}\quad & Y(i) = AN(i) && \text{[Production]} \\ & Y(i) = \left(\frac{P(i)}{P}\right)^{-\epsilon}C && \text{[Demand]}.\end{aligned}$$

The firm's Lagrangian is

$$\mathcal{L} = P(i)Y(i) - WN(i) + \lambda\left(AN(i) - Y(i)\right) + \gamma\left(\left(\frac{P(i)}{P}\right)^{-\epsilon}C - Y(i)\right).$$

The FOCs are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial Y(i)} &= P(i) - \lambda - \gamma = 0 && [C(i)] \\ \frac{\partial\mathcal{L}}{\partial N(i)} &= -W + \lambda A = 0 && [N(i)]\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial P(i)} = Y(i) - \gamma\epsilon \left(\frac{P(i)}{P}\right)^{-\epsilon-1} \frac{C}{P} = Y(i) - \gamma\epsilon \frac{Y(i)}{P(i)} = 0 \qquad [P(i)]$$

These give us

$$P(i) = \frac{\epsilon}{\epsilon-1}\frac{W}{A}.$$

This implies that $P(i)$ is constant across i, and then $P(i) = P, Y(i) = C, N(i) = N$.

9.2.3 Market Clearing

The goods market and labor market clear.
Government Budget Constraint:

$$\tau W N = T.$$

9.2.4 Equilibrium

The equilibrium is characterized by the following equations

$$\begin{aligned} N^{\varphi}C^{\sigma} &= (1-\tau)\frac{W}{P} \\ P &= \frac{\epsilon}{\epsilon-1}\frac{W}{A} \\ C &= AN. \end{aligned}$$

We can choose to normalize everything wrt P so that $P = 1$.

By setting

$$\begin{aligned} (1-\tau)\frac{\epsilon-1}{\epsilon} &= 1, \\ \Rightarrow \quad \tau &= -\frac{1}{\epsilon-1}, \end{aligned}$$

we can recover the social planner's problem.

10 New Keynesian Model: Dynamic

10.1 Setup

The CES consumption aggregator C is given by

$$C = \left(\int_{[0,1]} C(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

The ideal price index P for the CES aggregator is given by

$$P = \left(\int_{[0,1]} (P(i))^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$$

Note that P ensures that

$$\frac{P(i)}{P} = \left(\frac{C(i)}{C} \right)^{-\frac{1}{\epsilon}}, \quad PC = \int_{[0,1]} P(i)C(i)di.$$

10.1.1 Household

The household's preference is

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U \left(C_t, N_t, \frac{M_t}{P_t}; Z_t \right) \right].$$

A particular case of the utility function can be

$$U \left(C_t, N_t, \frac{M_t}{P_t} \right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \varrho v \left(\frac{M_t}{P_t} \right).$$

The household's budget constraint is given by

$$P_t C_t + Q_t B_t + M_t = (1-\tau) W_t N_t + D_t + T_t + B_{t-1} + M_{t-1}.$$

We can add Arrow securities as well.

Notation:

- B_t: Nominal bond that pays \$1 in the next period.
- Q_t: Price of nominal bond.
- M_t: Money holding.
- D_t: Dividends from firm.
  - Dividends are flows of income contingent on the state of world. We can price the ownership of firm (i.e. claims of dividends) without assuming Arrow securities. Because we have representative households, there won't be any trade among themselves. ??
- T_t: Lump sum transfer.

10.1.2 Firm

Each period, the firm may adjust its price with probability $(1-\theta)$, or is stuck with its previous price with probability θ. There is no physical cost of price adjustment otherwise.

The firm's problem conditional on "Calvo fairy" arriving is

$$\begin{aligned}
&\max_{P^*} \mathbb{E}_t \left[\sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left(P^* C_{t+k}(P^*) - W_{t+k} N_{t+k}(P^*) \right) \right] \\
\text{s.t.} \quad & C_{t+k}(P^*) = A_{t+k} N_{t+k}(P^*) \quad \text{[Technology]} \\
& C_{t+k}(P^*) = \left(\frac{P^*}{P_{t+k}} \right)^{-\epsilon} C_{t+k} \quad \text{[Demand]}
\end{aligned}$$

where $Q_{t,t+1}$ is the nominal SDF

$$\begin{aligned}
Q_{t,t+k} &\equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}} \frac{P_t}{P_{t+k}} \\
&= \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}}.
\end{aligned}$$

Note that here the SDF is the version without Pr. The $\frac{P_k}{P_{t+k}}$ term converts future nominal value to current nominal value. Also, it can be more rigorously written as $Q_t(z^{t+k}|z^t)$ which may help understand its meaning.

- As we have seen in the RBC model, firms using SDF to discount income flow implies that households own the firms. The firms' manager cares about households' welfare, but households' themselves do not control the production plan of the firms. This might be like some sort of principle-agent setup.

10.1.3 Government and Technology Shocks

Government controls money supply M_t^s, sets T_t, B_t, τ. It's budget constraint is

$$T_t + B_{t-1} = \tau W_t N_t + Q_t B_t.$$

$\ln A_t$ follows mean zero AR(1) process:

$$\mathbb{E}[a_t] \equiv \mathbb{E}[\ln A_t] = 0.$$

10.1.4 Competitive Equilibrium

Given initial distribution of prices $\{P(i)_{-1}\}_1$, a competitive equilibrium is a sequence of prices and quantities such that

- Given the prices, the quantities solves household's utility maximization problem.
- Given the prices, the quantities solves firm's profit maximization problem.

- Markets clear.
  - Labor Market
$$N = \int N(i)di$$
  - Goods Market
$$C_t(i) = A_t N(i)$$
  - Money
$$M_t = M_t^s$$
  - Asset Market
    * By Recardian equivalence between lump sum taxes and risk free bonds, WLOG $B_t = 0$.
    * If we let household trade Arrow securities with itself and firms, we can WLOG let security trading be 0.

10.2 First Order Conditions

10.2.1 Household

The household's Lagrangian is

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t, \frac{M_t}{P_t}; Z_t\right) \right.$$
$$\left. + \sum_{t=0}^{\infty} \beta^t \lambda_t \big((1-\tau)W_t N_t + D_t + T_t + B_{t-1} + M_{t-1} - P_t C_t - Q_t B_t - M_t\big) \right].$$

Note that
- Because $\lambda_t \equiv \lambda_t(z^t)$ corresponds to each t and z^t, it does not really matter whether we write the budget constraint in or outside of the expectation. λ_t in the two cases will be different by a probability term. But it might be easier to work with if we put the budget constraint in the expectation.
  - Recall that the expectation is integral over ω that leads to different z^t.
  - If we write the budget constraint outside of the expectation, we will need to take integral over z^t as well. ??
- For a similar reason, we can add discounting factors WLOG.

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial C_t(z^t)} = \beta^t \left(U_{C,t} - \lambda_t P_t\right) = 0 \qquad [C_t]$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial N_t(z^t)} =& \beta^t \left(U_{N,t} - \lambda_t(1-\tau)W_t\right) = 0 && [N_t] \\
\frac{\partial \mathcal{L}}{\partial B_t(z^t)} =& \beta^t \left(-\lambda_t Q_t + \mathbb{E}_t\left[\beta\lambda_{t+1}\right]\right) = 0 && [B_t] \\
\frac{\partial \mathcal{L}}{\partial M_t(z^t)} =& \beta^t \left(\frac{U_{M,t}}{P_t} - \lambda_t + \mathbb{E}_t\left[\beta\lambda_{t+1}\right]\right) = 0 && [M_t]
\end{aligned}$$

Here we implicitly use some results related to the possibility of exchanging order of integral and differentiation, as well as calculus of variation.

Rearranging the terms gives us

$$\begin{aligned}
&(1-\tau)\frac{W_t}{P_t} = \frac{U_{N,t}}{U_{C,t}} = N_t^{\varphi} C_t^{\sigma} && [N_t] + [C_t] \\
&Q_t = \beta \mathbb{E}_t\left[\frac{U_{C,t+1}}{U_{C,t}} \frac{P_t}{P_{t+1}}\right] = \beta \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right] && [B_t] + [C_t] \\
&1 - Q_t = \frac{U_{M,t}}{U_{C,t}} && [M_t] + [B_t] + [C_t]
\end{aligned}$$

10.2.2 Firm

We can rewrite firm's problem as the following unconstrained problem:

$$\max_{P_t^*} \mathbb{E}_t\left[\sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k}\left(P_t^* - \frac{W_{t+k}}{A_{t+k}}\right)\right]$$

The FOCs is

$$\begin{aligned}
&\mathbb{E}_t\left[\sum_{k=0}^{\infty} \theta^k Q_{t,t+k} \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k}\left((1-\epsilon) + \epsilon\frac{W_{t+k}}{A_{t+k}}\frac{1}{P_t^*}\right)\right] = 0 \\
\Rightarrow \quad &\mathbb{E}_t\left[\sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k}\left((1-\epsilon) + \epsilon\frac{W_{t+k}}{A_{t+k}}\frac{1}{P_t^*}\right)\right] = 0.
\end{aligned}$$

10.2.3 Additional Equilibrium Conditions

The aggregate price index is given by

$$P_t = \left(\int P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}},$$

where

$$P_t(i) = \begin{cases} P_t^* & \text{Calvo Fairy w.p. } 1-\theta \\ P_{t-1}(i) & \text{otherwise w.p. } \theta \end{cases}.$$

Market Clearing

- Labor Market

$$N_t = \int N_t(i)di$$

- Goods Market

$$C_t(i) = A_t N_t(i)$$

The two market clearing conditions and $C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} C_t$ then imply that

$$N_t = \frac{C_t}{A_t} \int \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di.$$

10.2.4 Money vs Interest Rate Rules

Money M_t only shows up in

$$1 - Q_t = \frac{U_{M,t}}{U_{C,t}}.$$

Any equilibrium where the government chooses a stochastic process $\{M_t\}$ corresponds to some equilibrium where it chooses stochastic process $\{Q_t\}$ and then use the previous equation to pin down M_t.

- Essentially for any given consumption path, only one of M_t and Q_t is free.

Therefore, we can think of equilibrium for a given stochastic process $\{Q_t\}$ and drop $\{M_t\}$ from analysis.

We define the following variables related to prices and interest rate

- Inflation (price change)

$$\Pi_t = \frac{P_t}{P_{t-1}}$$

- Nominal Interest Rate

$$I_t = \frac{1}{Q_t}$$

- Real Interest Rate

$$R_t = \frac{I_t}{\Pi_t}$$

We may also define the following variables for convenience.

$$\mu = \ln\left(\frac{\epsilon}{\epsilon - 1}\right)$$
$$\rho = -\ln(\beta)$$

10.3 Log-Linearization

10.3.1 Steady State in Deterministic Economy

We solve equilibrium using a version of perturbation technique. In particular, we log-linearize the system of differential equations around an equilibrium in a deterministic economy where:

- $a_t = 0$: i.e. deterministic;
- $P(i)_{-1} = P_{-1}$: degenerate prices;
- Tax set at efficient level, which means

$$(1-\tau)\frac{\epsilon-1}{\epsilon} = 1.$$

We can verify (by checking the differential equations are satisfied) that the following constitute an equilibrium (the one that we expand the system around):

- $P_t(i) = P_t = P_{-1}$; $P_t^* = P_{-1}$.
- $W_t = P_{-1}$.
- $Q_t = \beta$.
- $C_t(i) = C_t = 1$, $N_t(i) = N_t = 1$.

10.3.2 Useful Formula

The Taylor expansion formula for multivariate case is given by

$$T(x_1, \ldots, x_d) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1-a_1)^{n_1} \cdots (x_d-a_d)^{n_d}}{n_1! \cdots n_d!} \left(\frac{\partial^{n_1+\cdots+n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}}\right)(a_1, \ldots, a_d).$$

The first order expansion would be

$$T(\mathbf{x}) \approx T(\mathbf{a}) + \sum_{j=1}^{d} T_j(\mathbf{a})(x_j - a_j).$$

We make use of the following log-linearization formula

$$\begin{aligned} F(X) &= F(e^x) \\ &\approx F(e^{\overline{x}}) + F'(e^{\overline{x}})e^{\overline{x}}(x - \overline{x}) \\ &= F(\overline{X}) + F'(\overline{X})\overline{X}(x - \overline{x}). \end{aligned}$$

When X is a vector, we have

$$F(X) = F(\overline{X}) + \sum_{j=1}^{d} F'(\overline{X})\overline{X}\frac{\partial X}{\partial X_j}\bigg|_{X=\overline{X}}(x_j - \overline{x}_j).$$

A common way to log-linearize a ratio around 1 is

$$\frac{X_{t+1}}{X_t} = e^{\ln X_{t+1} - \ln X_t} \approx 1 + \ln X_{t+1} - \ln X_t.$$

10.4 Equilibrium: Planner's Problem

In the real economy, the social planner essentially solves a static problem. The FOCs are the same as in the previous chapter:

$$\begin{aligned} N_t^{\varphi} C_t^{\sigma} &= A_t \\ C_t &= A_t N_t. \end{aligned}$$

In the nominal economy, the implied real interest rate can be derived from the CE's Euler equation (the intertemporal choice equation)

$$\begin{aligned} Q_t &= \beta \mathbb{E}_t \left[\frac{U_{C,t+1}}{U_{C,t}} \frac{P_t}{P_{t+1}} \right] \\ \Rightarrow \; U_{C,t} &= \beta R_t \mathbb{E}_t \left[U_{C,t+1} \right], \end{aligned}$$

where we used $\Pi_t = 1$. When the utility function takes the specific form above, this expression is

$$C_t^{-\sigma} = \beta R_t \mathrm{E}_t \left[C_{t+1}^{-\sigma} \right]$$

10.4.1 Log-Linearization

Log-linearize the above equations, we have

$$\begin{aligned} \varphi n_t + \sigma c_t &= a_t \\ c_t &= a_t + n_t. \end{aligned}$$

We can solve for this system and get

$$n_t = \frac{1-\sigma}{\varphi + \sigma} a_t, \quad c_t = \frac{1+\varphi}{\varphi+\sigma} a_t.$$

- Note that a_t is exogenously given, so this specifies a solution to the SP problem.

Steady State
When $a_t = 0$ (in the steady state), we have

$$\overline{n}^e = \overline{c}^n = 0,$$

which implies $U_{C,t} = U_{C,t+1}$ and with the Euler equation

$$\beta \overline{R}^e = 1 \quad \Rightarrow \quad \overline{r}^e = -\ln \beta \equiv \rho.$$

Around Steady State
We conduct Taylor expansion for the Euler equation to gain more insight around the steady state. We can do the following to expand the equation around $r_t = \overline{r}^e = \rho$ and $\Delta c_{t+1} = 0$.

$$\begin{aligned}
1 =& \mathbb{E}_t \left[\beta R_t \frac{U_{C,t+1}}{U_{C,t}} \right] \\
\Rightarrow \quad 1 =& \mathbb{E}_t \left[\exp\left(\rho + r_t - \sigma(c_{t+1} - c_t) \right) \right] \\
\approx& \mathbb{E}_t \left[e^{\rho + \overline{r}^e - \sigma(\overline{c}^e - \overline{c}^e)} \right] \\
&+ \mathbb{E}_t [1] \, e^{\rho + \overline{r}^e - \sigma(\overline{c}^e - \overline{c}^e)} \left((\rho + r_t - \sigma(c_{t+1} - c_t)) - (\rho + \overline{r}^e - \sigma(\overline{c}^e - \overline{c}^e)) \right) \\
=& 1 + \mathbb{E}_t \left[r_t - \overline{r}^e - \sigma(c_{t+1} - c_t) \right] \\
\Rightarrow \quad c_t^e =& \mathbb{E}_t \left[c_{t+1}^e - \frac{1}{\sigma}(r_t^e - \rho) \right].
\end{aligned}$$

10.5 Equilibrium: Flexible Prices $\theta = 0$

Household's FOCs are

$$\begin{aligned}
(1-\tau)\frac{W_t}{P_t} &= N_t^{\varphi} C_t^{\sigma} \\
Q_t &= \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right].
\end{aligned}$$

Firm's FOCs now simplify to

$$\begin{aligned}
&\mathbb{E}_t \left[\left(\frac{C_{t+0}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+0}} \left(\frac{P_t^*}{P_{t+0}} \right)^{-\epsilon} C_{t+0} \left((1-\epsilon) + \epsilon \frac{W_{t+0}}{A_{t+0}} \frac{1}{P_t^*} \right) \right] = 0 \\
\Rightarrow \quad & (1-\epsilon) + \epsilon \frac{W_t}{A_t} \frac{1}{P_t^*} = 0,
\end{aligned}$$

which implies that all firms set price to the same value. (To get the second line, we take terms out of the expectation and then cancel out some of them; P_t^* cannot be 0 which causes demand to be unbounded.)

The aggregate price now coincides with firm optimal price

$$P_t = P_t^*.$$

Because $P_t(i)$ are equalized, $C_t(i)$ and $N_t(i)$ are also equalized across i. The market clearing conditions now boil down to a feasibility constraint

$$C_t = A_t N_t.$$

10.5.1 Log-Linearization

In this section we consider the case when $(1-\tau)\frac{\epsilon-1}{\epsilon} = 1$—that is, when flexible price equilibrium coincides with efficient allocations.

Real Variables
Following the same argument as for the SP problem, we have

$$n_t = \frac{1-\sigma}{\varphi+\sigma}a_t, \quad c_t = \frac{1+\varphi}{\varphi+\sigma}a_t.$$

This implies that the real variables are completely determined by the shocks and are not affected by the nominal variables.

Nominal Variables
The nominal variables are characterized by

$$\begin{aligned} c_t =& \mathrm{E}_t\left[c_{t+1} - \frac{1}{\sigma}(i_t - \pi_{t+1} - \rho)\right], \\ \pi_t =& \Delta w_t - \Delta a_t. \end{aligned}$$

Proof. We shall now log-linearize the expression for Q_t.

- Similar to the SP problem, we may expand the system around the "steady state" where $a_t = 0$. We can guess and verify that $i_t = \rho, \Delta c_{t+1} = 0, \pi_{t+1} = 0$ constitute the steady state.
- We can also do it slightly differently by following Jingoo's approach. In the steady state, we have

$$1 = \mathrm{E}_t\left[e^{-\rho+i-\sigma\Delta c-\pi}\right] \quad \Rightarrow \quad \rho = i - \sigma\Delta c - \pi.$$

Thus, we have

$$\begin{aligned} 1 =& \mathrm{E}_t\left[\frac{\beta}{Q_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{\Pi_{t+1}}\right] \\ =& \mathrm{E}_t\left[e^{-\rho+i_t-\sigma(c_{t+1}-c_t)-\pi_{t+1}}\right] \\ \approx& \mathrm{E}_t\left[e^{-\rho+i-\sigma\Delta c-\pi}\right] \\ &+ \mathrm{E}_t\left[e^{-\rho+i-\sigma\Delta c-\pi}(i_t - i)\right] \\ &- \mathrm{E}_t\left[e^{-\rho+i-\sigma\Delta c-\pi}\sigma(\Delta c_{t+1} - \Delta c)\right] \\ &- \mathrm{E}_t\left[e^{-\rho+i-\sigma\Delta c-\pi}(\pi_{t+1} - \pi)\right] \\ =& \mathrm{E}_t\left[1 + i_t - \sigma(c_{t+1} - c_t) - \pi_{t+1} - (i - \sigma\Delta c - \pi)\right] \\ =& \mathrm{E}_t\left[1 + i_t - \sigma(c_{t+1} - c_t) - \pi_{t+1} - \rho\right] \end{aligned}$$

$$\Rightarrow \quad c_t = \mathrm{E}_t \left[c_{t+1} - \frac{1}{\sigma}(i_t - \pi_{t+1} - \rho) \right].$$

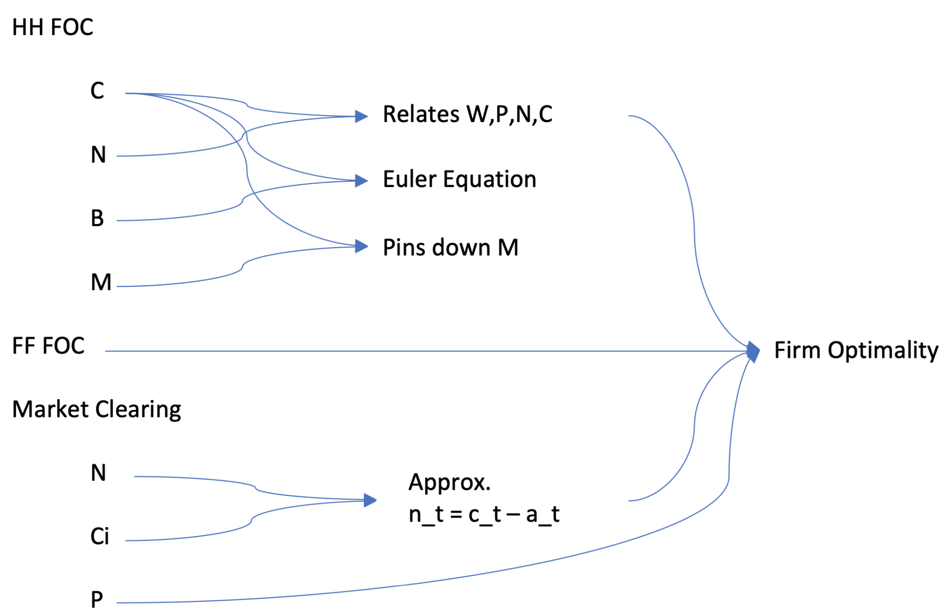
Finally, the firm's FOC gives us

$$\begin{aligned} p_t &= \ln\left(\frac{\epsilon}{\epsilon - 1}\right) + w_t - a_t \\ \Rightarrow \quad \pi_t &= \Delta w_t - \Delta a_t, \end{aligned}$$

where the second line is simply the first difference. □

10.6 Equilibrium: Sticky Prices $\theta > 0$

Figure 1: Roadmap for Derivation



10.6.1 Price Dynamics

The aggregate price index is given by

$$\begin{aligned} P_t &= \left(\int P_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}} \\ &= \left((1-\theta)(P_t^*)^{1-\epsilon} + \theta \int P_{t-1}(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}} \end{aligned}$$

$$= \left((1-\theta)(P_t^*)^{1-\epsilon} + \theta(P_{t-1})^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}.$$

It can be log-linearized as

$$\pi_t = (1-\theta)(p_t^* - p_{t-1}).$$

Proof. Dividing both sides by P_{t-1}, we can rewrite the price index as

$$\Pi_t^{1-\epsilon} = (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon} + \theta.$$

Notice that we have $\pi_t = 0, p_t^* = p_t$ in the steady state where $a_t = 0$, we can then use $e^x \approx 1 + x$ (for x close to 0) to approximate the system as

$$\begin{aligned} e^{(1-\epsilon)\pi_t} =& (1-\theta)e^{(1-\epsilon)(p_t^* - p_{t-1})} + \theta \\ \rightsquigarrow \quad 1 + (1-\epsilon)\pi_t \approx& (1-\theta)(1 + (1-\epsilon)(p_t^* - p_{t-1})) + \theta \end{aligned}$$

which then gives us what we want. □

10.6.2 Market Clearing

Total labor demand is not given by

$$\begin{aligned} N_t = \int N_t(i)di =& \int \frac{C_t(i)}{A_t}di \\ =& \frac{C_t}{A_t}\int \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di. \end{aligned}$$

This gives us

$$n_t = c_t - a_t. \tag{10.1}$$

Proof. Taking log of the equation above, we have

$$n_t = c_t - a_t + \ln\left(\int \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} di\right).$$

Thus, it would suffice to show that the last term is approximately 0.

By definition of the price index, we have

$$\begin{aligned} P_t^{1-\epsilon} =& \int P_t(i)^{1-\epsilon}di \\ \Rightarrow \quad 1 =& \int \left(\frac{P_t(i)}{P_t}\right)^{1-\epsilon} di \end{aligned}$$

$$
\begin{aligned}
&= \int e^{(1-\epsilon)(p_t(i)-p_t)} di \\
&\approx \int (1 + (1-\epsilon)(p_t(i) - p_t))\, di \\
\Rightarrow \quad & \int (p_t(i) - p_t) di \approx 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} di = & \int e^{(-\epsilon)(p_t(i)-p_t)} di \\
\approx & \int (1 - \epsilon(p_t(i) - p_t))\, di \\
\approx & 1,
\end{aligned}
$$

which shows the last term in the equation above is approximately 0. □

10.6.3 Household's FOCs

Household's FOCs remain the same as in the flexible price equilibrium. The Euler Equation is still

$$
c_t = \mathrm{E}_t \left[c_{t+1} - \frac{1}{\sigma}(i_t - \pi_{t+1} - \rho) \right].
$$

10.6.4 Firm's FOCs

Firm's FOCs is given by

$$
\mathbb{E}_t \left[\sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} C_{t+k} \left((1-\epsilon) + \epsilon \frac{W_{t+k}}{A_{t+k}} \frac{1}{P_t^*} \right) \right] = 0,
$$

which is more complicated in this case.

Eventually, our goal is to approximate it as

$$
\pi_t = \beta \mathrm{E}_t [\pi_{t+1}] + \lambda [(\varphi + \sigma) c_t - (1 + \varphi) a_t],
$$

where

$$
\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}
$$

Step 1: Rewrite with MC

We can first rewrite it as

$$\mathbb{E}_t \left[\sum_{k=0}^{\infty} (\theta\beta)^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} C_{t+k} \left(\frac{P_t^*}{P_{t-1}} + \frac{\epsilon}{\epsilon - 1} \frac{W_{t+k}}{A_{t+k} P_{t+k}} \frac{P_{t+k}}{P_{t-1}} \right) \right] = 0,$$

and we denote MC_{t+k} as

$$MC_{t+k} \equiv \frac{W_{t+k}}{A_{t+k} P_{t+k}}.$$

In the steady state, this FOC gives us

$$\frac{\epsilon}{\epsilon - 1} \overline{MC} = 1.$$

Log-linearize the equation around steady state, we have

$$p_t^* - p_{t-1} = (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\hat{mc}_{t+k} + (p_{t+k} - p_{t-1}) \right].$$

Step 2: Rewrite with π

We will try to rewrite the infinite sum involving price difference:

$$\begin{aligned}
\sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[p_{t+k} - p_{t-1} \right] &= \sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[\sum_{l=0}^{k} (p_{t+l} - p_{t+l-1}) \right] \\
&= \sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[\sum_{l=0}^{k} \pi_{t+l} \right] \\
&= \sum_{l=0}^{\infty} (\beta\theta)^l \left(\sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[\pi_{t+l} \right] \right) \\
&= \frac{1}{1 - \beta\theta} \sum_{l=0}^{\infty} (\beta\theta)^l \mathrm{E}_t \left[\pi_{t+l} \right]
\end{aligned}$$

Therefore, the firm's problem can be further reduce to

$$p_t^* - p_{t-1} = (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\hat{mc}_{t+k} \right] + \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \left[\pi_{t+k} \right].$$

Step 3: Recursive Formulation

Writing the equation from step 2 for $t+1$, taking expectation with respect to information at t, and applying LIE, we have

$$\begin{aligned}
\mathrm{E}_t\left[p^*_{t+1}-p_t\right] =& \mathrm{E}_t\left[(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathrm{E}_{t+1}\left[\hat{mc}_{t+1+k}\right]+\sum_{k=0}^{\infty}(\beta\theta)^k\mathrm{E}_{t+1}\left[\pi_{t+1+k}\right]\right]\\
=&(1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathrm{E}_{t}\left[\hat{mc}_{t+1+k}\right]+\sum_{k=0}^{\infty}(\beta\theta)^k\mathrm{E}_{t}\left[\pi_{t+1+k}\right]\\
=&\frac{1}{\beta\theta}(p^*_t-p_{t-1}-(1-\beta\theta)\mathrm{E}_t\left[\hat{mc}_t\right]-\mathrm{E}_t\left[\pi_t\right])\\
=&\frac{1}{\beta\theta}(p^*_t-p_{t-1}-(1-\beta\theta)\hat{mc}_t-\pi_t)
\end{aligned}$$

Therefore, we can write the equation recursively as

$$p^*_t-p_{t-1}=\beta\theta\mathrm{E}_t\left[p^*_{t+1}-p_t\right]+(1-\beta\theta)\hat{mc}_t+\pi_t.$$

Combining with the Price Dynamic Equation **??**, we have

$$\pi_t=\beta\mathbb{E}_t\left[\pi_{t+1}\right]+\underbrace{\frac{(1-\theta)(1-\beta\theta)}{\theta}}_{\equiv\lambda}\hat{mc}_t.$$

- Equivalently, we have in the sequential form

$$\pi_t=\lambda\sum_{k=0}^{\infty}\beta^k\mathrm{E}_t\left[\hat{mc}_{t+k}\right].$$

Step 4: Get Rid of MC

By definition of $\hat{mc}_{t+k}$, we have

$$\begin{aligned}
\hat{mc}_{t+k} =&(w_{t+k}-a_{t+k}-p_{t+k})-(w-a-p)\\
=&(w_{t+k}-a_{t+k}-p_{t+k})-\ln\left(\frac{\epsilon-1}{\epsilon}\right)\\
=&w_{t+k}-a_{t+k}-p_{t+k}+\ln(1-\tau)\\
=&(\ln(1-\tau)+w_{t+k}-p_{t+k})-a_{t+k}\\
=&\varphi n_{t+k}+\sigma c_{t+k}-a_{t+k}\\
=&\varphi(c_{t+k}-a_{t+k})+\sigma c_{t+k}-a_{t+k}\\
=&(\varphi+\sigma)c_{t+k}-(1+\varphi)a_{t+k}.
\end{aligned}$$

Eventually, we write the firm's FOC as

$$\pi_t=\beta\mathbb{E}_t\left[\pi_{t+1}\right]+\lambda\left((\varphi+\sigma)c_t-(1+\varphi)a_t\right).$$

10.6.5 Characterizing the Equilibrium

The log-linearized system is characterized by the following equations:

- Firm Optimality:

$$\pi_t = \beta \mathbb{E}_t [\pi_{t+1}] + \lambda ((\varphi + \sigma) c_t - (1 + \varphi) a_t) .$$

- Euler Equation:

$$c_t = \mathrm{E}_t \left[c_{t+1} - \frac{1}{\sigma} (i_t - \pi_{t+1} - \rho) \right] .$$

- Government controls i, and these two equations pin down c, π. Thus, the third equation that we need for the classical 3 equation NK model is some type of rule for how government sets i, e.g.

$$i_t = \rho + \phi \pi_t + v_t,$$

  where v_t is "monetary policy shock", ϕ is a coefficient describing how Central Bank adjusts interet rates in response to inflation.

  - Monetary policy affects real variables.
  - Once you know c, π, can back out n, w from other equations.

$$c_t = a_t + n_t, \quad \ln(1 - \tau) + w_t - p_t = \varphi n_t + \sigma c_t.$$

10.6.6 Solving the Model

10.6.7 Monetary Shocks

One-Time Monetary Shock at t

One-Time Monetary Shock at $t + 1$

One-Time Monetary Shock at $t + T$

10.6.8 TFP Schocks

One-Time TFP Shock at t

11 Appendix: Additional Materials

11.1 CES Demand (Discrete)

The following note make use of Acemoglu p.152 and p.423.

Consider the CES aggregator (Dixit-Stiglitz aggregator) of the form

$$c = \left(\sum_i \omega_i^{\frac{1}{\sigma}} (c_i + \bar{c}_i)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

This can be considered CES because if we redefine $\hat{c}_i = c_i + \bar{c}_i$, then the elasticity of substitution between any two goods $\hat{c}_i$ and $\hat{c}_j$ is equal to σ.

Consider the following utility maximization problem:

$$\max_{c_i, y} U(c, y) \text{ s.t. } \sum_i p_i(\hat{c}_i - \bar{c}_i) + y = m,$$

where y is an outside goods and serves as the numeraire.

We can write the Lagrangian as

$$\mathcal{L} = U(c, y) + \lambda(m - y + \sum_i p_i \bar{c}_i - \sum_i p_i \hat{c}_i).$$

The FOCs are

$$\frac{\partial \mathcal{L}}{\partial \hat{c}_i} = U_c(c, y)\omega_i^{\frac{1}{\sigma}} \left(\frac{\hat{c}_i}{c} \right)^{-\frac{1}{\sigma}} - \lambda p_i = 0,$$
$$\frac{\partial \mathcal{L}}{\partial c_i} = U_y(c, y) - \lambda = 0.$$

Combining the FOC for any i and j, we have

$$\frac{p_i}{p_j} = \left(\frac{\omega_i}{\omega_j} \right)^{\frac{1}{\sigma}} \left(\frac{\hat{c}_i}{\hat{c}_j} \right)^{-\frac{1}{\sigma}}.$$

11.1.1 Ideal Price Index

We define the ideal price index such that the following condition holds:

$$\frac{p_i}{p} = \omega_i^{\frac{1}{\sigma}} \left(\frac{\hat{c}_i}{c} \right)^{-\frac{1}{\sigma}}.$$

Note that this condition also ensures that

$$\sum_i \frac{p_i \hat{c}_i}{pc} = \sum_i \omega_i^{\frac{1}{\sigma}} \left(\frac{\hat{c}_i}{c}\right)^{1-\frac{1}{\sigma}} = c^{-\frac{\sigma-1}{\sigma}} \sum_i \omega_i^{\frac{1}{\sigma}} \hat{c}_i^{\frac{\sigma-1}{\sigma}} = c^{-\frac{\sigma-1}{\sigma}} c^{\frac{\sigma-1}{\sigma}} = c^0 = 1.$$

By writing $\hat{c}_i$ as an expression of the other variables and substituting it into the aggregator, we have

$$c = \left(\sum_i \omega_i^{\frac{1}{\sigma}} \left(\omega_i c \left(\frac{p_i}{p}\right)^{-\sigma}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

Simplifying this term gives us the price aggregator p as

$$p = \left(\sum_i \omega_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

In many circumstances, it is convenient to choose this ideal price index as the numeraire.

These also allow us to write the demand for $\hat{c}_i$ given c and prices as

$$\hat{c}_i = c\omega_i \left(\frac{p_i}{p}\right)^{-\sigma}.$$

11.2 Arrow Security

The competitive equilibrium in the Arrow economy is a collection of prices $\{Q_0(s_1), Q_t(s^t, s_{t+1})\}$ and quantities $\{c_t^i(s^t), A_t^i(s^t, s_{t+1})\}$ such that the following conditions are satisfied.

- Given the prices, the quantities solve consumers' utility maximization problem. The utility maximization problem for consumer i is

$$\begin{aligned}
&\max_{c^i,a^i} \sum_{t=1}^{\infty} \sum_{s^t} \beta^{t-1} \Pr[s^t] U(c_t^i(s^t)) \\
\text{s.t.} \quad &\sum_{s\in\mathcal{S}} Q_0(s) a_0^i(s) = 0 \\
&P_t c_t^i(s^t) + \sum_{s\in\mathcal{S}} Q_t(s^t, s) a_t^i(s^t, s) = P_t e_t^i(s^t) + a_{t-1}^i(s^{t-1}, s_t) \quad \forall t, s^t.
\end{aligned}$$

- Markets clear.
  - Goods market
$$\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \quad \forall t, s^t.$$
  - Asset market
$$\sum_{i=1}^{I} a_t^i(s^t, s_{t+1}) = 0 \quad \forall t, s^t, s_{t+1}.$$

11.2.1 Consumer's FOCs

The Lagrangian in the Arrow economy is

$$\begin{aligned}
\hat{\mathcal{L}} = &\sum_{t=1}^{\infty} \sum_{s^t} \beta^{t-1} \Pr[s^t] U(c_t^i(s^t)) - \hat{\eta}_0 \sum_{s} Q_0(\varnothing, s) A_0^i(\varnothing, s) \\
&+ \sum_{t=1}^{\infty} \sum_{s^t} \hat{\eta}_{t,s^t} \left(P_t e_t^i(s^t) + A_{t-1}^i(s^{t-1}, s_t) - P_t c_t^i(s^t) - \sum_{s\in\mathcal{S}} Q_t(s^t, s) A_t^i(s^t, s) \right).
\end{aligned}$$

The FOCs are

$$\begin{aligned}
\frac{\partial \hat{\mathcal{L}}}{\partial c_t^i(s^t)} &= \Pr[s^t] U'(c_t^i(s^t)) - \hat{\eta}_{t,s^t} P_t = 0 \\
\frac{\partial \hat{\mathcal{L}}}{\partial A_{t-1}^i(s^{t-1}, s_t)} &= \hat{\eta}_{t,s^t} - \hat{\eta}_{t-1,s^{t-1}} Q_{t-1}(s^{t-1}, s_t) = 0,
\end{aligned}$$

and the two sets of budget constraints.

From the FOCs, we have

$$\hat{\eta}_{t,s^t} = \hat{\eta}_{t-1,s^{t-1}} Q_{t-1}(s^{t-1}, s_t) = \hat{\eta}_0 Q_0(\varnothing, s_1) \prod_{k=2}^{t} Q_{k-1}(s^{k-1}, s_k), \tag{11.1}$$

and

$$Q_{t-1}(s^{t-1}, s_t) = \beta \frac{\Pr[s^t] U'(c_t^i(s^t))}{\Pr[s^{t-1}] U'(c_{t-1}^i(s^{t-1}))} \frac{P_{t-1}}{P_t}. \tag{11.2}$$