Theory of Income II
Note on Lecture 10 Preliminary

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2022-02-28

We now consider the competitive equilibrium with complete market in a neoclassical growth model without growth but with uncertainty. Let $s^t = (s_0, s_1, \ldots, s_t)$ denote the history of all shocks in the economy. We assume that $\Pr(s_0) = 1$ so there is no uncertainty in the initial period. The TFP shock $\ln Z_t$ is AR(1) and the technology is CRS.

1 Arrow-Debreu Economy

We will first set up the sequential problem in an Arrow-Debreu economy, which might be easier to think about.

1.1 Household's Problem

The representative household solves the maximization problem:

$$\max_{C,K,A} \sum_t \sum_{s^t} \beta^t \Pr(s^t) U(C_t(s^t))$$

$$\text{s.t.} \quad C_0(s^0) + K_1(s^0) + \sum_{t=1} \sum_{s^t} M(s^t)A(s^t) \leq W_0(s^0) + (1 + R_0(s^0) - \delta)K_0 + D_t(s^0),$$

$$C_t(s^t) + K_{t+1}(s^t) \leq W_t(s^t) + A(s^t) + (1 + R_t(s^t) - \delta)K_t(s^{t-1}) + D_t(s^t) \quad \forall t \geq 1, s^t.$$

- Prices and Dividends
  - $M(s^t)$ denote the price for Arrow-Debreu security $A(s^t)$ (in terms of period 0 consumption) that pays 1 unit of consumption good in the state s^t. Trade of Arrow-Debreu security only takes place in period 0.
    - * You could also assume that trade of Arrow-Debreu security takes place in period -1 before everything, then you will have the constraint $\sum_{t=0} \sum_{s^t} M(s^t)A(s^t) = 0$ as in Mike's slide.
  - $W_t(s^t)$ and $R_t(s^t)$ denote the wage and rental rate of capital in period t in terms of consumption $C_t(s^t)$. Writing the problem this way, we implicitly assume that other trade takes places in a sequence of spot markets.

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  * We can express these in terms of initial period consumption $C_0(s^0)$ as

$$W_t^0(s_t) = M(s^t)W_t(s_t), \quad R_t^0(s_t) = M(s^t)R_t(s_t).$$

- $D_t(s^t)$ denote the dividends from firm in period t in terms of consumption $C_t(s^t)$. Note that because the product and factor markets are competitive and we have CRS technology, $D_t(s^t) \equiv 0$.

- We can consolidate the constraints into one by multiplying each of them with $M(s^t)$ and summing up across t (there might be some implicit assumption about sums not being infinite):

$$\begin{aligned}&\sum_{t=0}\sum_{s^t} M(s^t)\left(C_t(s^t) + K_{t+1}(s^t)\right) \\ &\qquad \leq \sum_{t=0}\sum_{s^t} M(s^t)\left(W_t(s^t) + (1 + R_t(s^t) - \delta)K_t(s^{t-1}) + D_t(s^t)\right).\end{aligned}$$

  The reason that this is equivalent to the original set of constraints above is that for any values of $\{C_t(s^t), K_{t+1}(s^t), W_t(s^t), R_t(s^t), M(s^t)\}$ that satisfy this relationship, we can find a sequence of $A(s^t)$ such that the original constraints hold.

The consumer's Lagrangian is

$$\begin{aligned}\mathcal{L} = &\sum_t\sum_{s^t} \beta^t \Pr(s^t)U(C_t(s^t)) \\ &+ \lambda\sum_t\sum_{s^t} M(s^t)\left(W_t(s^t) + (1 + R_t(s^t) - \delta)K_t(s^{t-1}) + D_t(s^t) - C_t(s^t) - K_{t+1}(s^t)\right).\end{aligned}$$

The FOCs are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_t(s^t)} =& \beta^t \Pr(s^t)U'(C_t(s^t)) - \lambda M(s^t) = 0, \\ \frac{\partial \mathcal{L}}{\partial K_{t+1}(s^t)} =& \lambda\left(\sum_{s_{t+1}} M(s^{t+1})(1 + R_t(s^{t+1}) - \delta)K_{t+1}(s^t) - M(s^t)\right) = 0.\end{aligned}$$

The FOCs imply that

$$\frac{M(s^{t+1})}{M(s^t)} = \beta\frac{\Pr\left(s^{t+1}\right)U'(C_{t+1}(s^{t+1}))}{\Pr\left(s^t\right)U'(C_t(s^t))}.$$

1.2 Firm's Problem

The representative firm solves a static problem in each state of world:

$$\max_{\hat{K}_t(s^t),\hat{N}_t(s^t)} Z_t(s^t)F(\hat{K}_t(s^t), \hat{N}_t(s^t)) - W_t(s^t)\hat{N}_t(s^t) - R_t(s^t)\hat{K}_t(s^t).$$

The firm's FOCs are

$$\begin{aligned}\frac{\partial \mathcal{L}_f}{\partial \hat{K}_t(s^t)} =& Z_t(s^t)F_K(\hat{K}_t(s^t), \hat{N}_t(s^t)) - R_t(s^t) = 0, \\ \frac{\partial \mathcal{L}_f}{\partial \hat{L}_t(s^t)} =& Z_t(s^t)F_L(\hat{K}_t(s^t), \hat{N}_t(s^t)) - W_t(s^t) = 0.\end{aligned}$$

1.2.1 Alternative Setup for Firm

When the firm has some non-trivial inter-temporal choice to make, we will need to set up the firm's problem differently. We will illustrate this type of argument in our current setup even though our firm faces essentially a static problem.

We make some implicit assumptions in this alternative setup.

- The firm is owned by the household. From the household's perspective, a firm is nothing but an asset that pays some state-contingent dividends.
- The household can choose to trade firm's ownership in period 0 before the dividends are distributed (or in period -1).[1]
- Therefore, we can use prices of AD securities to price the value of the firm, which is

$$\sum_t \sum_{s^t} M(s^t) \underbrace{\left(Z_t(s^t) F(\hat{K}_t(s^t), \hat{N}_t(s^t)) - W_t(s^t)\hat{N}_t(s^t) - R_t(s^t)\hat{K}_t(s^t) \right)}_{\equiv D_t(s^t)}.$$

  The firm will maximize this value.

Therefore, the firm's optimization problem is

$$\max_{\hat{K}_t(s^t), \hat{N}_t(s^t)} \sum_{s^t} M(s^t) D_t(s^t).$$

The firm's FOCs are

$$\begin{aligned}
\frac{\partial \mathcal{L}_f}{\partial \hat{K}_t(s^t)} =& M(s^t) \left(Z_t(s^t) F_K(\hat{K}_t(s^t), \hat{N}_t(s^t)) - R_t(s^t) \right) = 0, \\
\frac{\partial \mathcal{L}_f}{\partial \hat{L}_t(s^t)} =& M(s^t) \left(Z_t(s^t) F_L(\hat{K}_t(s^t), \hat{N}_t(s^t)) - W_t(s^t) \right) = 0.
\end{aligned}$$

We can see from these FOCs that the firm's problem is essentially static, as the optimal choice only involves prices from the given state s^t. This shows us why assuming the firm simply solves a static problem would be equivalent.

1.3 Competitive Equilibrium in AD Economy

The competitive equilibrium is a collection of prices and quantities such that

1. Given the prices, the quantities solve the household's optimization problem.
2. Given the prices, the quantities solve the firm's optimization problem.
3. All markets clear:
   - Labor market:

$$\hat{N}_t(s^t) = N_t(s^t) \equiv 1$$

[1] We can have other equivalent setup for trading in firm's ownership (e.g. allowing for trading in every period), and the formal proof for the equivalence should be similar to the proof for the equivalence of AD and A economy.

- Capital market:

$$\hat{K}_{t+1}(s^t) = K_{t+1}(s^t)$$

- Asset market:

$$A(s^t) = 0.$$

- Good market (feasibility):

$$C_t(s^t) = Z_t(s^t)F(K_{t+1}(s^t), N_t(s^t)) + (1-\delta)K_t(s^{t-1}) - K_{t+1}(s^t).$$

2 Arrow Economy

To write up the problem recursively, it would be helpful to formulate the same sequential problem in the Arrow economy.

2.1 Household's Problem

The representative household solves the maximization problem:

$$\begin{aligned}
&\max_{C,K,A} \sum_t \sum_{s^t} \beta^t \Pr(s^t) U(C_t(s^t)) \\
\text{s.t.} \quad & C_t(s^t) + K_{t+1}(s^t) + \sum_{s_{t+1}} M_t(s_{t+1}|s^t) A(s_{t+1}|s^t) \\
& \leq W_t(s^t) + A(s_t|s^{t-1}) + (1 + R_t(s^t) - \delta) K_t(s^{t-1}) + D_t(s^t) \quad \forall t, s^t.
\end{aligned}$$

- $M_t(s_{t+1}|s^t)$ denote the price for Arrow security $A(s_{t+1}|s^t)$ that pays 1 unit of consumption good in the state (s^t, s_{t+1}). $A(s_0|s^{-1}) = 0$ so the agents are not endowed with any security in the very beginning.
- To rule out Ponzi schemes, we need to impose borrowing constraints on the household's asset position. To simplify our discussion, we do not go into details here, but it suffices to assume that there is a arbitrary debt limit of 0 and show that in the equilibrium the constraint is not binding.

The consumer's Lagrangian is

$$\begin{aligned}
\mathcal{L} = & \sum_t \sum_{s^t} \beta^t \Pr(s^t) U(C_t(s^t)) \\
& + \sum_t \sum_{s^t} \eta(s^t) \Bigg(W_t(s^t) + A(s_t|s^{t-1}) + (1 + R_t(s^t) - \delta) K_t(s^{t-1}) + D_t(s^t) \\
& \qquad C_t(s^t) - K_{t+1}(s^t) - \sum_{s_{t+1}} M_t(s_{t+1}|s^t) A(s_{t+1}|s^t) \Bigg).
\end{aligned}$$

The FOCs are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_t(s^t)} &= \beta^t \Pr(s^t) U'(C_t(s^t)) - \eta(s^t) = 0, \\
\frac{\partial \mathcal{L}}{\partial K_{t+1}(s^t)} &= \sum_{s_{t+1}} \eta(s^{t+1})(1 + R_t(s^{t+1}) - \delta) K_{t+1}(s^t) - \eta(s^t) = 0, \\
\frac{\partial \mathcal{L}}{\partial A(s_{t+1}|s^t)} &= \eta(s^{t+1}) - \eta(s^t) M_t(s_{t+1}|s^t) = 0.
\end{aligned}$$

The FOCs imply that

$$M_t(s_{t+1}|s^t) = \beta \frac{\Pr\left(s^{t+1}\right) U'(C_{t+1}(s^{t+1}))}{\Pr\left(s^t\right) U'(C_t(s^t))}.$$

Using a similar equivalence argument as in Problem Set 5, we will have

$$M_t(s_{t+1}|s^t) = \frac{M(s^{t+1})}{M(s^t)}$$

which can be alternatively written as

$$M(s^{t+1}) = \prod_{i=0}^{t} M_i(s_{i+1}|s^i)$$

2.2 Firm's Problem

The representative firm's problem is the same as before.

2.3 Competitive Equilibrium in A Economy

The conditions are the same, except the the asset market clearing condition is now

$$A(s_{t+1}|s^t) = 0.$$

3 Recursive Competitive Equilibrium

For a sequential problem, we should be able to write a corresponding recursive problem. The solution to the sequential problem and that to the recursive problem would coincide under certain conditions. For instance, Nancy's RMED provides some such conditions, which Mike thinks are often too strong/restrictive. Nonetheless, in this course, we will operate under the assumption that the two formulations do lead to the same solution. In practice, people also often operate under such assumptions unless your research question is closely related to the equivalence.

We can define a recursive competitive equilibrium as a set of functions that satisfies a set of conditions. This is similar to our definition of a sequential competitive equilibrium, which, as you all know by heart, is a set of prices and quantities (that are essentially functions of time or history) that satisfies a set of conditions. The functions in the definition of a recursive competitive equilibrium typically include value functions of agents that have non-trivial inter-temporal decisions, policy functions, price functions, and law of motion of some aggregate state variables. The arguments of these functions are aggregate or individual state variables. The conditions that these functions have to satisfy typically include agents' Bellman equation, agents' optimality conditions, market clearing conditions, and a condition that the actual law of motion implied by the policy functions coincide with the law of motion of the state variables that we specify (this last condition reflects the rational expectation hypothesis). You can also find an example without uncertainty in RMED Chapter 16.

In our example, the set of functions that defines the recursive equilibrium are as follows. We will describe the full definition and the conditions that they have to satisfy after we go through the agents' problems and market clearing conditions. For convenience of notation, define

$$x = (k, a), \quad X = (K, A, Z).$$ [2]

- $V(x, X)$: Value function of the household.
- $\tilde{c}(x, X), \tilde{k}_+(x, X), \tilde{a}_+(Z_+|x, X)$: Policy functions of the household.
  - $\tilde{a}_+(Z_+|x, X)$ denotes the amount of Arrow security that yields one unit of consumption in the next period if the state Z_+ is realized. Think of Z_+ as an index for the Arrow security.
- $D(X)$: Firm's profit and dividend to the household.
- $\tilde{n}^d(X), \tilde{k}^d(X)$: Policy functions of the firm.
- $\tilde{W}(X), \tilde{M}(X_+|X), \tilde{R}(X)$: Price functions.
- $\tilde{K}_+(X), \tilde{A}_+(Z_+|X)$: Law of motion for aggregate states.

[2] We do not really need A as an aggregate state, which we will show later, but we include it here as if we have heterogeneous agent, in which case we would likely need to track the distribution of Arrow securities. Think of A as the distribution of Arrow securities here, not the aggregate quantity of Arrow securities.

3.1 Household's Problem

You have set up so many Bellman equations in Nancy's class so you should be familiar with this. What might be slightly different here is that the household takes the price functions and aggregate law of motion as given when making their decision. This kind of setup also appeared in Nancy's lecture on RBC without uncertainty.

The household's Bellman equation is given by

$$V(k,a,K,A,Z) = \max_{c,k_+,a_+(Z_+)} U(c) + \beta \sum_{Z_+} \Pr(Z_+|Z)\, V(k_+, a_+(Z_+), \tilde{K}_+(X), \tilde{A}_+(Z_+|X), Z_+)$$

$$\text{s.t.}\quad c + k_+ + \sum_{Z_+} \tilde{M}(\tilde{K}_+(X), Z_+|X) a(Z_+) \leq \tilde{W}(X) + (1 + \tilde{R}(X) - \delta)k + a + D(X).$$

3.2 Firm's Problem

The firm's problem is static:

$$D(X) = \max_{k^d,n^d} ZF(k^d, n^d) - W(X)k^d - R(X)n^d.$$

The optimality conditions are then:

$$ZF_K(k^d, n^d) = R(X), \quad ZF_L(k^d, n^d) = W(X)$$

3.3 Market Clearing

In the following conditions, we are being very pedagogical by writing out the aggregation explicitly.

- Aggregations:

$$\begin{aligned} &K^d = k^d, \quad K = k, \\ &N^d = n^d, \quad N = 1, \\ &A = a, \\ &A_+(Z_+) = a_+(Z_+), \\ &C = c, \quad K_+ = k_+. \end{aligned}$$

  Note that these aggregations come from our representative agent assumption. If agents are heterogeneous, the aggregate quantities will then be given by $\int y dF(y)$ where $F(y)$ is the CDF for the distribution of y. Among these aggregations, $K = k$ is particularly important, which will become clear in our definition of recursive competitive equilibrium.

- Labor Market:

$$N^d = N$$

- Capital Market:

$$K^d = K$$

- Asset Market:

$$A_+(Z_+) = 0$$

- Consumption goods

$$C + K_+ = ZF(K^d, N^d) + (1-\delta)K.$$

3.4 Definition of Recursive Competitive Equilibrium

A recursive competitive equilibrium in this economy is a set of functions $\{V, \tilde{c}, \tilde{k}_+, \tilde{a}_+, \tilde{D}, \tilde{n}^d, \tilde{k}^d, \tilde{W}, \tilde{M}, \tilde{R}, \tilde{K}_+, \tilde{A}_+\}$ such that

- V satisfies the Bellman equation for the household, and $\tilde{c}, \tilde{k}_+, \tilde{a}_+$ are the associated optimal policy functions.
- $\tilde{n}^d$ and $\tilde{k}^d$ satisfy the firm's optimality conditions and imply profit $D(X)$.
- Markets clear. We shall replace the lower case variables in the aggregation equations with the policy functions.
- The implied LOM coincides with the LOM that we specify:

$$\begin{aligned} \tilde{A}_+(Z_+|X) =& A_+(Z_+) = 0, \\ \tilde{K}_+(K, Z) =& K_+ = \tilde{k}(K, 0, K, 0, Z). \end{aligned}$$

  Note that we also impose that the aggregate supply of Arrow security is 0, so $A = a = 0$. We can see from here that we do not really need A (and a) in the state variable because they are always constant 0. [FIXME: Double check this argument.]

However, you may have noticed that many of these equations are simply accounting identities. If we combine different things, we will have a more succinct version of the definition.

A recursive competitive equilibrium is a set of functions $\{V, \tilde{c}, \tilde{k}_+, \tilde{W}, \tilde{R}, \tilde{K}_+\}$ such that

- V satisfies the Bellman equation for the household, and $\tilde{c}, \tilde{k}_+$ are the associated optimal policy functions:

$$\begin{aligned} V(k, K, Z) =& \max_{c, k_+,} U(c) + \beta \sum_{Z_+} \Pr(Z_+|Z)\, V(k_+, \tilde{K}_+(K, Z), Z_+) \\ \text{s.t.} \quad & c + k_+ \leq \tilde{W}(K, Z) + (1 + \tilde{R}(K, Z) - \delta)k. \end{aligned}$$

- The firm's optimality conditions are satisfied:

$$ZF_K(K, 1) = R(K, Z), \quad ZF_L(K, 1) = W(K, Z).$$

- The implied LOM coincides with the LOM that we specify:

$$\tilde{K}_+(K, Z) = \tilde{k}_+(K, K, Z).$$

3.5 More Discussion Related to the State Variables

[Forthcoming]