TOI 1 TA Session 2
Welfare Theorems

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Logistics

- Please make sure that you check the suggested solutions for problem sets if you feel unsure about anything when you work through them.
  - Our grading of your problem sets will not reflect all potential issues, and our grading of your exams will be more stringent.
- As we said before, we all know that you have access to past solutions, but it is extremely important that you understand the materials and do not blindly copy those (plus, there could be errors and it can be embarrassing if you copy those as well).
- When typesetting, please start a new problem on a new page.

Introduction

Some Definitions

Competitive Equilibrium (CE)

- A competitive equilibrium is a set of prices and quantities, such that, given prices, the quantities solve agents' optimization problems, and all markets clear. (All agents see the same prices.)
    - A decentralized equilibrium or competitive equilibrium with taxes is the same as a competitive equilibrium, except that different agents may see different effective prices.

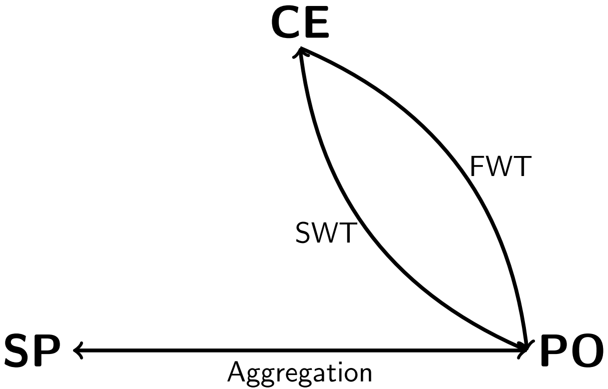
Pareto Optimal Allocation (PO)

- An allocation is Pareto efficient if there does not exist another feasible allocation such that all agents are at least as well off, and at least one agent is strictly better off.

Social Planner (SP)

- A social planner solves an optimization problem subject to feasibility constraints, where the objective function being optimized is typically a weighted sum or average of individual agent's utility.

We will connect them with Welfare Theorems



Some Technical Definitions

An economy has the following components

- L: **Commodity space**.
- I: A set of **households** (indexing).
- $X^i \subseteq L$: **Consumption possibility set**
- $u^i : X^i \to \mathbb{R}$: **Utility function**
- J: A set of **firms** (indexing).
- $Y^j \subseteq L$: **Technology**
- $\theta^i_j \geq 0$: **Ownership**
- $e^i \in L$: **Endowment** of agent i

First Welfare Theorem

Definition of Feasible Allocations

Definition

A feasible allocation $\{x^i, y^i\}$ satisfies three conditions:

1. Each consumer can consume x^i:

$$x^i \in X^i \quad \forall i \in I.$$

2. Each firm can produce y^j:

$$y^j \in Y^j \quad \forall j \in J.$$

3. Demand equals supply

$$\sum_{i \in I} x^i = \sum_{j \in J} y^j + \sum_{i \in I} e^i.$$

Note that we sometimes use the notation $\bar{e} = \sum_{i \in I} e^i$ since only the sum matters for feasible allocation.

Formal Definition of CE

Definition

We denote prices by a vector $p \in \mathbb{R}^{\dim L}$. A competitive equilibrium is a price vector, and a feasible allocation $\{x^i, y^j\}$ such that

1. Each firm $j \in J$ maximizes its profits π^j with y^j:

$$\pi^j \equiv \max_{y \in Y^j} p \cdot y$$

2. Each consumer maximize their utility subject to their budget constraint with x^i:

$$\max_{x \in X^i} u^i(x) \quad \text{s.t.} \quad p \cdot x \leq p \cdot e^i + \sum_{j \in J} \theta^i_j \pi^j.$$

The LHS of the budget constraint has the value of the net purchases of the households. The RHS of the budget constraint contains the source of funds for the net purchases.

Formal Definition of PO

Definition

A feasible allocation $\{\overline{x}^i, \overline{y}^j\}$ is PO if there is no other feasible allocation $\{x^i, y^j\}$ preferred by everyone, i.e. one such that

$$u^i(x^i) \geq u^i(\overline{x}^i) \quad \forall i \in I$$
$$u^i(x^i) > u^i(\overline{x}^i) \quad \exists i \in I.$$

First Welfare Theorem

Assumption

Local Non-Satiation *(LNS)*
u^i, X^i satisfies local non-satiation if for any $x \in X^i$ and any neighborhood (open ball) of x, B_ϵ, there is an $\hat{x} \in B_\epsilon(x) \cap X^i$ such that $u^i(\hat{x}) > u^i(x)$.

Theorem

First Welfare Theorem
Assume that the preferences for all agents satisfies local non-satiation. Let $\{p, \overline{x}^i, \overline{y}^j\}$ be a CE. Then $\{\overline{x}^i, \overline{y}^j\}$ is a PO allocation.

Proof of FWT (1/4)

BWOC, assume that there is a feasible allocation $\{x^i, y^j\}$ that Pareto dominates $\{\overline{x}^i, \overline{y}^j\}$, i.e.

$$u^i(x^i) \geq u^i(\overline{x}^i) \text{ for all } i \in I \text{ and } u^{i'}(x^{i'}) > u^{i'}(\overline{x}^{i'}) \text{ for some } i' \in I.$$

Then, it must be that

$$px^i \geq p\overline{x}^i \text{ for all } i \in I \tag{1}$$

$$px^{i'} > p\overline{x}^{i'} \text{ for some } i' \in I. \tag{2}$$

Proof of FWT (2/4)

1. To see that (1) must hold, notice that if, by contradiction, $px^i < p\overline{x}^i$, then, by the local-non satiation assumption, there must be a $\hat{x} \in X_i$ in a neighborhood of x^i such that $u^i(\hat{x}) > u^i(x^i)$, and by choosing the neighborhood small enough, $p\hat{x} \leq p\overline{x}^i$. This will contradict that $\overline{x}^i$ maximizes utility, and hence (1) must hold.
2. To see that (2) must hold, notice that if, by contradiction, $px^{i'} \leq p\overline{x}^{i'}$, then $x^{i'}$ is budget-feasible, and hence it contradicts that $\overline{x}^{i'}$ solves the consumer problem.

Proof of FWT (3/4)

Then adding (1) across consumers and noticing that (2) holds with strict inequality for some consumer i', we have

$$\left.\begin{aligned} px^i &\geq p\bar{x}^i \quad \forall i \in I \\ px^{i'} &> p\bar{x}^{i'} \end{aligned}\right\} \Rightarrow$$

$$\sum_{i\in I} px^i > \sum_{i\in I} p\bar{x}^i \quad \Leftrightarrow \quad p\sum_{i\in I} x^i > p\sum_{i\in I} \bar{x}^i. \tag{3}$$

On the firm side, since $\bar{y}^j$ maximizes profits for all j, and y^j is feasible,

$$p\bar{y}^j \geq py^j \; \forall j \in J \quad \Rightarrow$$

$$\sum_{j\in J} p\bar{y}^j \geq \sum_{j\in J} py^j \quad \Leftrightarrow \quad p\sum_{j\in J} \bar{y}^j \geq p\sum_{j\in J} y^j. \tag{4}$$

Proof of FWT (4/4)

Since both $\{x^i, y^j\}$ and $\{\overline{x}^i, \overline{y}^j\}$ are feasible,

$$p\sum_{i\in I} x^i = p\sum_{j\in J} y^j + p\sum_{i\in I} e^i, \tag{5}$$

$$p\sum_{i\in I} \overline{x}^i = p\sum_{j\in J} \overline{y}^j + p\sum_{i\in I} e^i, \tag{6}$$

a contradiction with (3) and (4).

- To see this, recall that (3) implies LHS of (5) $>$ LHS of (6), and (4) implies RHS of (5) $\leq$ RHS of (6). Also notice that the remaining term is the same in (5) and (6).

$\square$

Comment on the Proof

The last step when we derive the contradiction is not as innocuous as it seems. The argument using the inequality would not work if we are comparing infinite to infinite (in TOI3, we will see this in an Overlapping Generation model). Therefore, an additional condition that ensures the FWT is that

$$\sum_{i \in I} p x_i < \infty.$$

Second Welfare Theorem

Assumptions for Second Welfare Theorem

Assumption

HH: X^i are convex for all i, and u^i are continuous and strictly quasi-concave.

- i.e. The upper contour sets of u

  $$\{x \in X^i : u^i(x) \geq u^i(\overline{x})\}$$

  are strictly convex for all i and all $\overline{x} \in X^i$.

FF: The aggregate production set of the economy Y is convex:

$$Y = \left\{ y \in L : y = \sum_{j=J} y^j, \text{ for some } y^j \in Y^j \ \forall j \in J \right\}.$$

Second Welfare Theorem (Main Part)

Theorem

Let $\{\overline{x}^i, \overline{y}^j\}$ be a PO allocation. Then there exists a price vector p such that

1. All firms maximize profits:

$$p\overline{y}^j \geq py \quad \forall y \in Y^j, \ \forall j \in J.$$

2. Given the allocation $\overline{x}^i$, consumers minimize expenditure subject to attaining at least the same utility obtained with $\overline{x}^i$:

$$\overline{x}^i \in \underset{x \in X^i}{\operatorname{argmax}}\, px \text{ s.t.} u^i(x) \geq u^i(\overline{x}^i).$$

Note that we still need some weak assumptions to turn the cost minimization problem into a utility maximization problem (via dual problem). Arrow's remark provides a sufficient condition.

Assumptions for Social Planner's Problem

Definition

Utility Possibility Set
The utility possibility set U is defined as the set of utilities that are achievable for a feasible allocation, i.e.

$$U = \{u \in \mathbb{R}^I : u^i \leq u^i(x^i)\ \forall i, \text{ for some feasible } \{x^i, y^j\}\}.$$

(u^i is the i-th element of u.)

Proposition

If the aggregate possibility set Y and all the consumption possibility sets X^i are convex, then the set of feasible allocations is convex.

Assumption

CC: u^i are concave for all $i \in I$.

Connecting SP and PO

Theorem

Assume that u^i are strictly increasing, and that assumptions HH, CC, and FF are satisfied.
Then $\{\overline{x}^i, \overline{y}^i\}$ is a PO allocation if and only if there is a vector $\lambda \in \mathbb{R}_+^I$ such that $\{\overline{x}^i, \overline{y}^i\}$ solves the problem W:

$$W : \quad \max_{\{x^i, y^j\}} \sum_{i \in I} \lambda_i u^i(x^i)$$

subject to $\{x^i, y^j\}$ being a feasible allocation.