TA Session 1: Linear City Model

Feng Lin

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1 Introduction

In this TA session, we will take a look at a linear city model from Bordeu (2023) (Job Market Paper of Olivia Bordeu, who was advised by Professor Rossi-Hansberg). Some takeaways that we hope you may get from this TA session:

- Economists can add bells and whistles to the basic linear city model to illustrate the mechanism in more complicated spatial frameworks.
- Some intuition in the basic linear city model can carry over to more complicated ones.
- We do not expect you to understand all the technical details in this note, but hopefully it will give you a better sense of the crucial components in linear city models.

2 Model Setup

2.1 Location

Consider a linear city with a finite number of locations:

- $j \in \mathcal{J} = \{1, \ldots, J\}$: J distinctive locations along a line.
- $\tilde{L}_{Fj}$: Number of workers working in location j.
- $\tilde{L}_{Hj}$: Number of workers residing in location j.
- $A_j(\tilde{L}_{Fj})$: Productivity in location j with functional form

$$A_j(\tilde{L}_{Fj}) = \overline{A}_j \tilde{L}_{Fj}^{\gamma_F}.$$

- $B_j(\tilde{L}_{Rj})$: Amenities in location j with functional form

$$B_j(\tilde{L}_{Rj}) = \overline{B}_j \tilde{L}_{Rj}^{\gamma_R}.$$

- $\overline{H}_{Fj}$: Fixed supply of land for production purposes in location j.
- $\overline{H}_{Rj}$: Fixed supply of land for residential purposes in location j.

Comment
Some noticeable features in comparison to the basic linear city model:

- A main deviation from the basic linear city model is to allow for production in all locations within the city. Arguably this is a more realistic assumption and allows for analysis of the distribution of production activities within a city.
- Limited land supply for production adds another congestion force so that production will not be concentrated at one single location.

2.2 Firm

- A representative firm in location j produce a freely traded goods using Cobb-Douglas technology

$$Y_j = A_j(\tilde{L}_{Fj}) \left(\frac{L_{Fj}}{\alpha}\right)^{\alpha} \left(\frac{H_{Fj}}{1-\alpha}\right)^{1-\alpha}$$

- All markets (product, labor, land) are competitive.
- The firm's optimization problem is:

$$\max_{L_j, H_{Fj}} A_j(\tilde{L}_{Fj}) \left(\frac{L_{Fj}}{\alpha}\right)^{\alpha} \left(\frac{H_{Fj}}{1-\alpha}\right)^{1-\alpha} - W_j L_{Fj} - Q_{Fj} H_{Fj}.$$

Comment
Some noticeable features in comparison to the basic linear city model:

- As mentioned above, production is now possible in all locations and firms using land in production is another congestion force in the model.
- Other than these, the production side is not very different from the basic linear city model as it still features a CRS technology and competitive markets (leading to zero profit in the equilibrium).

2.2.1 Firm's Optimality Conditions

The FOCs are

$$\frac{\partial \Pi}{\partial L_{Fj}} = A_j(\tilde{L}_{Fj}) \left(\frac{L_{Fj}}{\alpha}\right)^{\alpha-1} \left(\frac{H_{Fj}}{1-\alpha}\right)^{1-\alpha} - W_j = 0,$$

$$\frac{\partial \Pi}{\partial H_{Fj}} = A_j(\tilde{L}_{Fj}) \left(\frac{L_{Fj}}{\alpha}\right)^{\alpha} \left(\frac{H_{Fj}}{1-\alpha}\right)^{-\alpha} - Q_{Fj} = 0.$$

This gives us the standard Cobb-Douglas result:

$$\frac{W_j L_{Fj} j}{Q_{Fj} H_{F_j}} = \frac{\alpha}{1-\alpha}.$$

In equilibrium, we can related $W_j, L_{Fj}, \overline{H}_{Fj}$ as

$$W_j = A_j(L_{Fj}) \left(\frac{\alpha}{1-\alpha} \frac{\overline{H}_{Fj}}{L_{Fj}}\right)^{1-\alpha}$$

- In equilibrium $\tilde{L}_{Fj} = L_{Fj}$ and $H_{Fj} = \overline{H}_{Fj}$.
- We get the equation above by substituting out Q_{Fj} in the second FOC. We can use the Cobb-Douglas result to derive Q_{Fj} with the other variables.

2.2.2 Firm Internalizing Externalities

Suppose that the firm now interalizes the production externalities. The FOCs are

$$\begin{aligned} \frac{\partial \Pi}{\partial L_{Fj}} &= \overline{A}_j \frac{\gamma_F + \alpha}{\alpha^{\alpha}} L_{Fj}^{\gamma_F + \alpha - 1} \left(\frac{H_{Fj}}{1-\alpha}\right)^{1-\alpha} - W_j = 0, \\ \frac{\partial \Pi}{\partial H_{Fj}} &= \overline{A}_j L_{Fj}^{\gamma_F} \left(\frac{L_{Fj}}{\alpha}\right)^{\alpha} \left(\frac{H_{Fj}}{1-\alpha}\right)^{-\alpha} - Q_{Fj} = 0. \end{aligned}$$

This gives us an equation similar to the standard Cobb-Douglas result:

$$\frac{W_j L_{Fj}}{Q_{Fj} H_{F_j}} = \frac{\gamma_F + \alpha}{1-\alpha}.$$

In equilibrium, we can related $W_j, L_j, \overline{H}_{Fj}$ as

$$W_j = \frac{\gamma_F + \alpha}{\alpha} A_j(L_{Fj}) \left(\frac{\alpha}{1-\alpha} \frac{\overline{H}_{Fj}}{L_{Fj}}\right)^{1-\alpha}.$$

2.3 Worker

- Workers make the following choices (simultaneously):
  1. Whether to move to the city.
  2. Where to live and commute.

3. Expenditure on consumption and housing.

- Worker ν's utility function when residing in i, working in j, consuming good C and housing H is given by

$$U_{ij}(C,H,\nu) = \frac{B_i(\tilde{L}_{Ri})}{\tau_{ij}} \left(\frac{C}{\beta}\right)^{\beta} \left(\frac{H}{1-\beta}\right)^{1-\beta} \epsilon_{1ij}(\nu),$$

where τ_{ij} is the equilibrium commuting cost from i to j and $\epsilon_{1ij}(\nu)$ represents their idiosyncratic preferences.

  - The idiosyncratic preference is drawn from a Generalized Extreme Value distribution

$$G(\{\epsilon_{cij}\}) = \exp\left(-\sum_{c\in\{0,1\}} \left[\sum_{ij\in\mathcal{J}^2} \epsilon_{cij}^{-\theta}\right]^{-\frac{\mu}{\theta}}\right)$$

  $c = 1$ denotes living in the city and $c = 0$ denotes lliving outside of the city.

Comment

Some noticeable features in comparison to the basic linear city model:

- Modeling commuting costs as a utility term is a standard trick in modern frameworks. It should be possible to show that it is fundamentally similar to some monetary costs given certain preference structure.
- Allowing workers to choose the amount of housing disentangle the equivalence between land area and population. This allows for somewhat more realistic behavior of workers (e.g. workers would demand fewer units of housing in more expensive locations).
- The idiosyncratic shocks in preference is not very consequential in affecting the main message of the model. Typically, they may help model match data better or allow for easier convergence in simulations.

2.3.1 Worker's Optimality Conditions

- Conditional on residing in i and working in j, the worker's problem is

$$\max_{C,H} \frac{B_j(\tilde{L}_{Rj})}{\tau_{ij}} \left(\frac{C}{\beta}\right)^{\beta} \left(\frac{H}{1-\beta}\right)^{1-\beta} \quad \text{s.t.} \quad C + Q_{Ri}H = W_j$$

  The FOC gives

$$\frac{C}{Q_{Ri}H} = \frac{\beta}{1-\beta}.$$

This implies that

$$\begin{aligned} C_{ij} =& \beta W_j \\ Q_{Ri} H_{ij} =& (1-\beta) W_j . \end{aligned}$$

- Conditional on residing in i and working in j, worker ν's indirect utility function is given by

$$V_{ij}(\nu) = \underbrace{\frac{B_i(\tilde{L}_{Ri})}{\tau_{ij}} \frac{W_j}{Q_{Ri}^{1-\beta}}}_{\equiv V_{ij}} \epsilon_{1ij}(\nu).$$

- The number of workers residing in i and working in j is related to the number of workers living in city by

$$L_{ij} = \frac{V_{ij}^{\theta}}{\sum_{od} V_{od}^{\theta}} L.$$

- The number of workers living in city is related to the total number of workers by

$$L = \frac{U^{\mu}}{U^{\mu} + \overline{U}^{\mu}} \overline{L},$$

where $\overline{L}$ denotes the total number of workers in the economy and

$$U = \left(\sum_{ij} V_{ij}^{\theta} \right)^{\frac{1}{\theta}}$$

denotes the expected utility of workers living in the city.

2.4 Commute

- The equilibrium commute cost τ_{ij} depends on the number of workers that make use of the common route.
- Workers commute between τ_{ij} uses all edges kl between i and j.
  - Indicator function $\mathbb{1}_{ij}^{kl} = 1$ if kl is between i and j.
- Let M_{kl} denote the number of workers that make use of edge kl:

$$M_{kl} = \sum_{ij} L_{ij} \mathbb{1}_{ij}^{kl}.$$

- Commute cost τ_{ij} is given by

$$\tau_{ij} = \prod_{kl} \exp\left(\kappa \bar{t}_{kl} M_{kl}^{\sigma}\right) \mathbb{1}_{ij}^{kl},$$

where κ controls disutility from commuting, $\bar{t}_{kl}$ denotes some edge-specific characteristics affecting commute time, and σ denotes congestion elasticity.

Comment

Some noticeable features in comparison to the basic linear city model:

- We open the black box of commute cost here to some extent by linking it to certain features of the network and commute flows (instead of assuming it is simply a function of distance).
- Bordeu (2023) analyzes how government fragmentation affect the spatial distribution economic activities through infrastructure that main affects commute. Therefore, she needs a better characterization of this part of the model.

2.5 Other Components

- Land is owned by absentee landlords who derive utility from the consumption of the traded good.

- By residential land market clearing, we have

$$\overline{H}_{Ri} = \sum_{j} L_{ij} H_{ij} = \sum_{j} L_{ij}(1-\beta)W_j.$$

- We have implicitly used production land market clearing in the expression that relates $W_j, L_j, \overline{H}_{Fj}$.

3 Solve the Model

Given model parameters, we want to solve for $\{W_j, Q_{Fj}, Q_{Rj}, L_{Fj}, L_{Rj}, L_{ij}, M_{kl}, \tau_{ij}, C_{ij}, H_{ij}, U, L\}$.

The key unknowns variables (loosely defined as the ones that can be used to derive other variables with very simple equations) are:

- W_j: J unknowns.
- Q_{Ri}: J unknowns.
- L_{ij}: $J \times J$ unknowns.
- L: 1 unknown.

The set of equations that characterizes these unknowns are:

$$W_j = A_j(L_{Fj}) \left(\frac{\alpha}{1-\alpha} \frac{\overline{H}_{Fj}}{L_{Fj}} \right)^{1-\alpha} \qquad J \tag{3.1}$$

$$\overline{H}_{Ri} = \sum_j L_{ij} H_{ij} \qquad J \tag{3.2}$$

$$L_{ij} = \frac{V_{ij}^{\theta}}{\sum_{od} V_{od}^{\theta}} L \qquad J \times J \tag{3.3}$$

$$L = \frac{U^{\mu}}{U^{\mu} + \overline{U}^{\mu}} \overline{L} \qquad 1 \tag{3.4}$$

where we have by definition/market clearing

$$\sum_i L_{ij} = L_{Fj} \qquad J$$

$$\sum_j L_{ij} = L_{Ri} \qquad J$$

$$Q_{Ri} H_{ij} = (1-\beta) W_j \qquad J \times J$$

$$M_{kl} = \sum_{ij} L_{ij} \mathbb{1}_{ij}^{kl} \qquad J-1$$

$$\tau_{ij} = \prod_{kl} \exp\left(\kappa \bar{t}_{kl} M_{kl}^{\sigma} \right) \mathbb{1}_{ij}^{kl} \qquad J \times J$$

$$V_{ij} = \frac{B_i(L_{Ri})}{\tau_{ij}} \frac{W_j}{Q_{Ri}^{1-\beta}} \qquad J \times J$$

$$U = \left(\sum_{ij} V_{ij}^{\theta} \right)^{\frac{1}{\theta}} \qquad 1$$

4 Simulation Results

We use the following parameters:

Table 1: Model Parameters

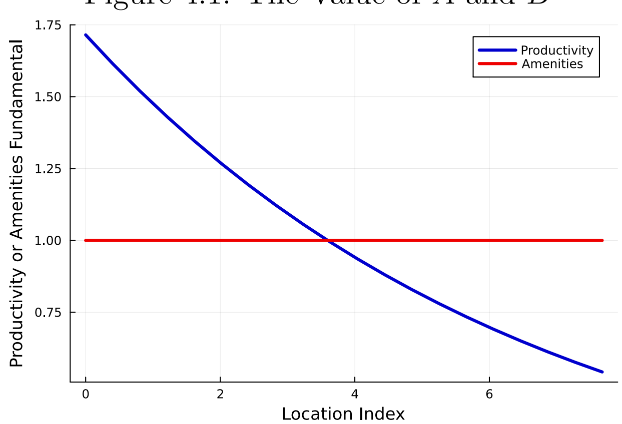
Parameter	Value
J	20
$\overline{A}$	See Below
γ_F	See Below
$\overline{B}$	See Below
γ_R	0
α	0.80
β	0.75
κ	0.008
t	1
σ	0.15
θ	7.0
μ	5.0
$\overline{U}$	1.0
$\overline{L}$	1000.0

We assume $\overline{A}$ and $\overline{B}$ to be the following for $j = 1, \dots, 20$

$$A_j = 100 \times \frac{e^{0.15\times(j-1)}}{\sum_i e^{0.15\times(i-1)}}, \quad B_j = 1.$$

Graphically, they are the following:

Figure 4.1: The Value of $\overline{A}$ and $\overline{B}$



4.1 With and Without Agglomeration

Figure 4.2: Employment or Population

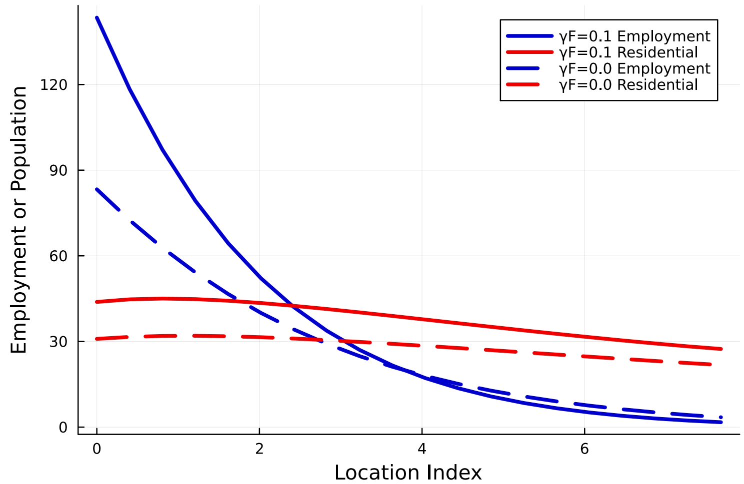
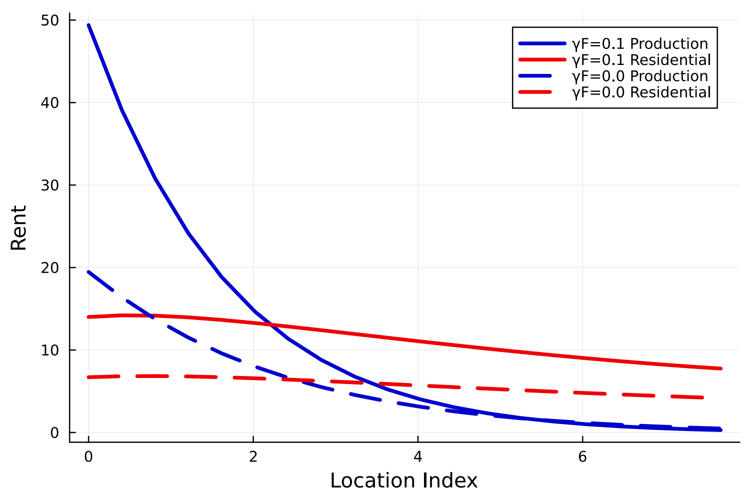


Figure 4.3: Rent

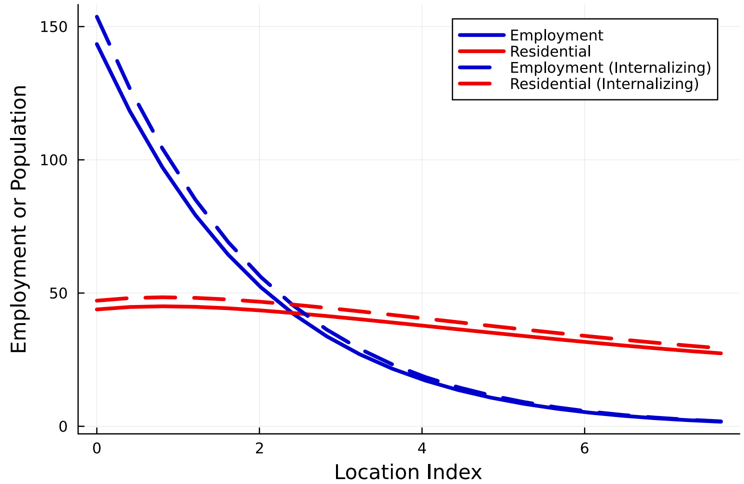


Comments

- With agglomeration, the more productive locations become even larger relative to the less productive locations.
- Even though all locations are the same in terms of amenities, population still concentrates in the city center because of better access to high-paying jobs.
- Similar to the basic linear city model, we also see rent to be higher in the city center. Locations further away have longer commutes to good jobs.

4.2 Internalizing Externalities

Figure 4.4: Employment or Population



Comments

- If firms internalize externalities, they hire more workers and the city becomes larger. The city under market economy is insufficiently small.

References

Bordeu, Olivia, "Commuting Infrastructure in Fragmented Cities," Working Paper 2023.