EC 232 TA Session 3
Linearization

Feng Lin

10/18/2021

Why Linearization?

- As we discussed before, many optimization problems that economists deal with can be characterized by a system of ODEs. However, it could be very difficult to find the global solution to these ODEs.
- What can we do?
  - First, we will focus on local properties of the solution; for instance, around the steady states.
  - Second, we will approximate general problems like $\dot{x} = f(x)$ with something that we know how to solve, first order linear ODEs, by linearizing the system.

Some Preparation

Taylor Expansion

Under proper conditions, the Taylor expansion for $f(x_1, \ldots, x_d)$ around $(a_1, \ldots, a_d)$ is given by

$$T(x_1, \ldots, x_d) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d}}{n_1! \cdots n_d!} \left(\frac{\partial^{n_1+\cdots+n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \right) (a_1, \ldots, a_d).$$

The first order expansion would be

$$T(x_1, \ldots, x_d) \approx T(\mathrm{a}) + \sum_{j=1}^{d} T_j(\mathrm{a})(x_j - a_j).$$

This is what we will use the most often when we do linearization.

Interpreting $\ln(X) - \ln(\overline{X})$

When X is close to $\overline{X}$, $\ln(X) - \ln(\overline{X})$ is approximately the percentage deviation of X from $\overline{X}$. To see this, consider the first order expansion of $\ln(X)$ around $\overline{X}$ using the first order Taylor expansion formula in the previous page:

$$\ln(X) \approx \ln(\overline{X}) + \frac{1}{\overline{X}}(X - \overline{X})$$

$$\Rightarrow \quad \ln(X) - \ln(\overline{X}) \approx \frac{X}{\overline{X}} - 1$$

Cookbook for Log-Linearization

The Basic Machinery

Consider a function $f(X)$. Let x denote $\ln(X)$.
A common log-linearization of $f(X)$ around $\overline{X}$ that will underlie many common cases is

$$\begin{aligned} f(X) &= f(e^x) \\ &\approx f(e^{\overline{x}}) + f'(e^{\overline{x}})e^{\overline{x}}(x - \overline{x}) \\ &= f(\overline{X}) + f'(\overline{X})\overline{X}(x - \overline{x}). \end{aligned}$$

In the multivariate case, we have

$$f(\mathrm{X}) \approx f(\overline{\mathrm{X}}) + \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}(\overline{\mathrm{X}})\overline{X}_j(x_j - \overline{x}_j).$$

Example: Expanding X around $\overline{X} = 1$

Implicitly, X can be seen as a function $f(X) = X$. This view allows us to directly apply the formula from the previous page

$$\begin{aligned} f(X) =& f(e^{\ln(X)}) \\ \approx& f(\overline{X}) + f'(\overline{X})\overline{X}(x - \overline{x}) \\ =& \overline{X} + 1 \times \overline{X} \times /(\ln(X) - \ln(\overline{X})) \\ =& 1 + 1 \times 1 \times (\ln(X) - \ln(1)) \\ =& 1 + \ln(X). \end{aligned}$$

A variation of this that is often used:

$$\begin{aligned} \frac{X_{t+1}}{X_t} =& e^{\ln X_{t+1} - \ln X_t} \\ \approx& 1 + \ln X_{t+1} - \ln X_t. \end{aligned}$$

Example: An Additive Function

Consider expanding

$$f(X) = AX^2 + BX + C$$

around $\overline{X}$.

$$f(X) \approx A\overline{X}^2 + B\overline{X} + C + (2A\overline{X} + B)\overline{X}(x - \overline{x})$$

Example: A Multiplicative Function

Consider a Cobb-Douglas production function:

$$Y = K^\alpha L^\beta.$$

Assume that we expand around $\overline{Y} = \overline{K}^\alpha \overline{L}^\beta$.

- For the LHS:

$$Y \approx \overline{Y}(1 + y - \overline{y}).$$

- For the RHS:

$$\begin{aligned} K^\alpha L^\beta \approx & \overline{K}^\alpha \overline{L}^\beta + \alpha \overline{K}^{\alpha-1} \overline{L}^\beta \overline{K}(k - \overline{k}) + \beta \overline{K}^\alpha \overline{L}^{\beta-1} \overline{L}(l - \overline{l}) \\ = & \overline{K}^\alpha \overline{L}^\beta (1 + \alpha(k - \overline{k}) + \beta(l - \overline{l})). \end{aligned}$$

- Combining the two, we have

$$y - \overline{y} = \alpha(k - \overline{k}) + \beta(l - \overline{l}).$$

Example: A Multiplicative Function (Cont.)

Note that in this very simple example we could actually derive the same equation by just taking logs and subtract the steady state values. But when the equation is more complicated, that trick may not work.