EC 232 TA Session 1

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Recap: Why Ordinary Differential Equations?

- In economics, we are almost always dealing with some (constrained) optimization problems. Often they have a time dimension.
- (Lecture 1 Slide 7) Solutions to these problems are
  1. optimal time paths for the control(s) and state(s), or
  2. optimal policy functions, that produce values for the control(s) as functions of the state(s).

  In the first few weeks of this class, the solutions are all time paths.
- We often derive a set of necessary and sufficient conditions to characterize the solution to a optimization problem. And these conditions will constitute a system of ODEs.

Solving ODEs

Solving ODEs

When solving ODEs, we are searching for unknown function that satisfies some given conditions involving its derivatives. For instance, let x be an unknown function of t. We are given a condition

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}\right) = 0,$$

and we are interested in finding x that satisfies this relationship.

In this class, we mostly only need to know how to solve the most basic homogeneous and autonomous ODEs in the form of

$$\dot{x} = ax.$$

More often, we want to characterize the solution without actually solving ODEs explicitly. We will talk about this more later.

Solving $\dot{x} = ax$

Let $a \in \mathbb{R}$. We are given that

$$\dot{x} = ax, \tag{LOM}$$
$$x(0) = x_0. \tag{Boundary Condition}$$

We can do the following:

$$\begin{aligned}
&\int \frac{\dot{x}}{x} dt = \int a dt \\
\Rightarrow &\int \frac{1}{x} \frac{dx}{dt} dt = \int a dt \\
\Rightarrow &\int \frac{1}{x} dx = \int a dt \\
\Rightarrow &\ln(x) + c = at \\
\Rightarrow &x(t) = e^{at-c} \equiv Ce^{at}.
\end{aligned}$$

We can then use the initial condition to pin down C.

Small Side Comment

Note that the solution to $\dot{x} = ax$ will never be 0 unless $x \equiv 0$. This may sometime turn out to be helpful to pin down the sign of certain variables.

Stability

Loosely speaking, we call a time path (i.e. a function of time) stable if it converges to some finite value in infinite time. Consider our example from above, where we have the solution

$$x(t) = e^{at-c}.$$

We have

$$\lim_{t\to\infty} x(t) = \begin{cases} 0 & \text{if } a < 0 \\ x_0 & \text{if } a = 0 \\ \infty & \text{if } a > 0 \end{cases}.$$

Thus, the system is unstable when $a > 0$.

System of ODEs

More often, the necessary and sufficient conditions characterizing the solution constitute a system of ODEs. As you may have noticed, we have a LOM for the state and another one for the costate, leading to a system of ODEs.

It is highly unlikely that you will ever be asked to solve a system of ODEs explicitly in this class (unless one of the LOM implies a constant state or costate, as in the Hotelling's model). However, it is still possible that you may be asked to verify that a guess is a solution or to characterize the solution (see below).

A Useful Trick: Guess and Verify

At times we for some reason could guess (correctly) the solution to a system of ODEs. If so, we would only need to check that the solution (e.g. time paths) indeed satisfies the necessary and sufficient conditions that characterize the system.

You have seen a working example of this in the Hotelling's model with competitive equilibrium (Lecture 1 Slide 35-36).

- Specifically, we conjecture that prices are the same as implied by the SP problem, and make a guess of the quantity sellers deliver. Note that the solution that we guess may not be unique, unless some conditions (that I don't really know) are satisfied.

You probably will see this method applied in later problem sets.

Be Careful with FOCs

When we write the Hamiltonian,

$$H(t, \mathrm{x}, \mathrm{u}, \boldsymbol{\lambda}) = f(t, x, u) + \boldsymbol{\lambda} \cdot \mathrm{g}(t, x, u), \tag{1.1}$$

the first "main" necessary (and sufficient) condition is

$$\mathrm{u}^* = \underset{\mathrm{u} \in U}{\operatorname{argmax}} \, H.$$

In practice, we often just write the derivative of H with respect to u. However, be careful with cases where some constraint is binding and we need to consider corner solutions. In such cases, we would not set $\frac{\partial H}{\partial u} = 0$, but may require it to be positive or negative based on monotonicity of the functions.

Phase Diagram

Characterizing Steady State and Dynamics

As we have mentioned repeatedly before, in this class you may be asked to characterize the solution of systems of ODEs without solving them. Here is an example (RCK Growth).

Consider the following system of ODEs:

$$\begin{aligned}\dot{k} =& f(k) - \delta k - c \\ \dot{c} =& (f'(k) - \delta - \rho)c\end{aligned}$$

where

- f is is twice continuously differentiable, strictly increasing, strictly concave, satisfies Inada conditions, and $f(0) = 0$.

We will follow the cookbook approach:

1. Write out the loci $\dot{c} = 0$ and $\dot{k} = 0$ (intersection is steady state).
2. Consider deviation from loci and determine how the variables flow.